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Comparison of Cylindrical and Conical Helical Springs for their Buckling Load and Deflection

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Abstract

Buckling behavior of cylindrical helical springs subjected to axial load has already been illustrated in the past theories. Unlike cylindrical spring, the behavior of conical compression spring has both linear and nonlinear phase. Very little literature on buckling of conical springs can be found in the past. In this paper, an analytical buckling equation with its experimental verification has been proposed by authors and used it along with the existing theories to locate the phase of compression of conical spring at which buckling occurs. Subsequently, a comparison between cylindrical and conical springs has been made at the point of buckling of cylindrical spring in respect of their load and deflection. This would help to decide the suitability of conical springs against buckling failure of cylindrical springs under the given operating conditions.

Keywords: *buckling behavior, nonlinear behavior, conical spring, critical load and deflection, cylindrical spring*

1. Introduction

The coil compression springs will have a tendency to buckle when the deflection (for a given free length) becomes too large and thereby spring can no longer provide the intended force. Buckling can be prevented by limiting the deflection or the free length of the spring. If these options are unavailable, then best alternative is replacing with conical springs(having variable radii of curvature) to avoid buckling as they have less likeliness to this instability. In most of the research completed to date on buckling of coiled springs, very few literature on conical springs can be found. In this paper, authors have proposed an analytical expression with its experimental verification along with existing bench mark theories to compare cylindrical and conical springs at the point of buckling of cylindrical spring in respect of their load and deflection. The prediction of buckling of conical compression springs having constant pitch using author's newly developed equation and that of E.B.Wolansky [8] shows the upper and lower bound values of buckling of conical springs between which all experimental data lies. This would help to decide the suitability of conical springs against buckling failure of cylindrical springs under the given operating conditions. It also, helps to predict the possibility of buckling of conical helical springs beforehand at the design stage.

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The deflection analysis has been carried out with the springs made of ASTM A313 Type 304 (Stainless steel) material and these theoretical values are experimentally verified.

2. Literature Review

A. M. Wahl [2] has summarized basic and essential definitions, characteristics, behavior models, and calculation methods, load-deflection equation relating to the main types of springs. According to Wahl's assumptions, the derivation is accurate for cases where deflections per coil in axial direction of the springs are not too large and pitch angles are less than 10^0. J. A. Haringx [3, 4], demonstrated that an excellent agreement exists between the experiments and the results of a new theoretical calculation with respect to the elastic stability of helical compression springs of circular wire section. This calculation showed that the critical relative compression at which buckling occurs depends only on the ratio of initial length, L_o to the coil diameter D_o and on the method of attaching the spring ends. Finally, a continuous relation has been derived for a nonlinear conical spring by Rodrigues *et al.*, [6]. They illustrated that the conical compression spring behavior has a linear phase but can also have a nonlinear phase. The rate of the linear phase can easily be calculated but no analytical model exists to describe the nonlinear phase precisely. This nonlinear phase can only be determined by a discretizing algorithm. They presented analytical continuous expressions of length as a function of load and vice versa for a constant pitch conical compression spring in the nonlinear phase. Validation of new conical spring models in comparison with experimental data is performed. The behavior law of a conical compression spring can now be analytically determined Becker *et al.*, (4, 13), partial differential equations governing the buckling behavior of helical compression sprngs were developed and solved for both end fixed and circular cross-section using transfer matrix method and produced buckling design charts. And the equations governing resonant frequencies of a helical spring subjected to a static axial compressive load are solved numerically using transfer matrix method for clamped ends. H. Wang *et al.*, [14], has developed load-deflection relationships by using strain-energy method and nonlinear effects due to compression of the large diameter coils have been discussed. M.H.Wu *et al.*, [9], has proposed a model to calculate load-deflection relation of the conical spring and verified experimentally with static data. It shows that the maximum error between simulation and experimental results was 4.6 %. V. Yildirim [11, 12], has developed free vibration equations for cylindrical isotropic helical springs loaded axially and solved numerically based on the transfer matrix method to perform buckling analysis in a dynamic manner. The axial and shear deformation effects together with rotator inertia effects are all considered based on the first order shear deformation theory. However, Wolansky.E.B [8], has derived the buckling –deflection equation of conical spring for both simply-supported and fixed ends. The deflection due to shear load is omitted, and only energy from torsional and flexural stresses were considered.

Therefore, based on this review of literature and research to date, the author has attempted to develop an analytical equation considering the effect of shear deformation also which was previously ignored and verify it experimentally so that a comparison may be made between cylindrical and conical springs at the point of buckling of cylindrical spring in respect of their load and deflection. This would help to decide the suitability of conical springs against buckling failure of cylindrical springs under the given operating conditions. It also, helps to predict the possibility of buckling of conical helical springs beforehand at the design stage.

3. Methodology

The following cylindrical helical springs whose specifications are given in the following Table 1 has been used for buckling analysis.

Table 1. Specifications of Cylindrical Springs Made ASTM A313 Type 304 (Stainless Steel)

Outer coil diameter (D_o), mm	Free Length of the Coil(L_f), mm	Mean coil diameter (D_m) mm	Wire diameter (d) mm	Pitch of the coil (p) mm	Helix angle (α_o)	Spring Index ($C=D_m/d$),	Total Number of turns (n_o)
20	100	18	2	5.04	5.1	9	20
20	105	18	2	5.23	5.30	9	20
20	126	18	2	5.04	5.10	9	25
20	147	18	2	5.60	5.70	9	25
20	175	18	2	6.73	6.80	9	26
20	190	18	2	7.33	7.40	9	26
20	200	18	2	7.68	7.74	9	27
20	230	18	2	8.83	8.80	9	27
20	252	18	2	9.64	9.73	9	28

Note: All springs have squared and ground ends.
Material properties taken from hand book [1978].
For ASTM A313 Type 304(SS)
$E = 1.93 \text{ x}10\text{^}5 \text{ N/mm}^2$; $G = 70.3 \text{ x } 10\text{^}3 \text{ N/mm}^2$; $\rho = 79200 \text{ N/m}^3$, $S_{ult} = 1717.5 \text{ N/ mm}^2$.

3.1.Theoretical Analysis for Cylindrical Helical Compression Springs

Much of the relations given by A. M. Wahl [2], J. A. Hanrigx [3] and L. E. Becker and W. L. Cleghorn [4] has been used.

For the purpose of buckling analysis, the following equation for straight coil close helical compression springs and other fundamental related expressions [2, 3] are used.

$$\xi = \frac{\delta_{cr}}{l_o} = 0.8125\left\{1 \pm \sqrt{1 - 6.87\left(\frac{D_o}{l_o}\right)^2}\right\} \quad (1)$$

where D_o = mean coil diameter.
Note:The above equation can also be used for **both end fixed** by taking,

$$l_o = \frac{l_o}{2} \quad (2)$$

And other related equations[3]:

i) Axial Pitch,

$$p = \tan\alpha_o \times \pi D_m \quad (3)$$

ii) Ratio,

$$\frac{L_f}{D_m} = n_0 \pi \tan \alpha_o \quad (4)$$

iii)) To determine the helix angel at buckling stage,

$$\frac{\delta_{cr}}{l_0} = 1 - \left(\frac{\sin\alpha}{\sin \alpha_0}\right) \quad (5)$$

iii) Ratio of turns.

$$\frac{n}{n_0} = \frac{\left(\frac{EI_y}{GI_w}\right)[\cos\alpha_o \cos \alpha + \sin\alpha_o \sin \alpha]}{\left(\frac{EI_y}{GI_w}\right)\cos^2\alpha + \sin^2\alpha} \quad (6)$$

iv) Critical load,

$$P_{cr} = \left(\frac{GI_w}{R^2}\right)\left[\sin\alpha_0 \left(\frac{n}{n_0}\right)^{-1} - \sin\alpha\right] \quad (7)$$

3.2. Numerical Illustration

For Spring No.3 from Table 1.

L_f = 126 mm, Do= 20 mm, d=2mm, D_m = 18 mm, p=5.04 mm, $\alpha_o = 5.1^0$, C = D_m/d = 9 mm, n_o=25, L_f / D_m= 7 mm, n = 23, S_{ult} = 1717.5 N/ mm^2 for Gr-II , E= 1.93 x 10^5 & G=70.3 x 10^3

i) $p = \tan\alpha_o \times \pi Dm$ or $L_f = n_o D_m \pi \tan\alpha_o$ =25 x 18 x π x $\tan\alpha_o$ $\therefore \alpha_o = 5.1^0$

ii) $\frac{L_f}{D_m} = 7\ mm$

iii) Relative Critical deflection,

$$\xi = \frac{\delta_{cr}}{l_o} = 0.8125\left\{1 \pm \sqrt{1 - 6.87\left(\frac{2\times 18}{126}\right)^2}\right\} = 0.274$$

$$\delta_{cr} = 0.274 \times 126 = 34.53\ mm$$

iv) To determine the helix angel at buckling stage,

$$0.274 = 1 - \left(\frac{\sin\alpha}{\sin 5.10}\right) \quad \therefore \alpha = 3.7^0$$

v) Ratio of number of turns, $\frac{n}{n_0} = 0.91$

vi) Critical Load, (P_{cr}) = $34\ N = 3.46\ Kg$

vii) Wahl's correction factor, $K_w = \frac{4\times 9-1}{4\times 9-4} + \frac{0.615}{9} \quad \therefore K_w = 1.162$

viii) Corrected torsional shear stress,

$$\tau' = 1.162 \times \left[\frac{8\times 34 \times 18}{\pi \times 2^3}\right] = 226.36\ N/mm^2$$

For safeness of the springs,

$$\tau' < 772. N/mm^2 \ (= 0.45 \times S_{ult} = 0.45 \times 1717.5)$$

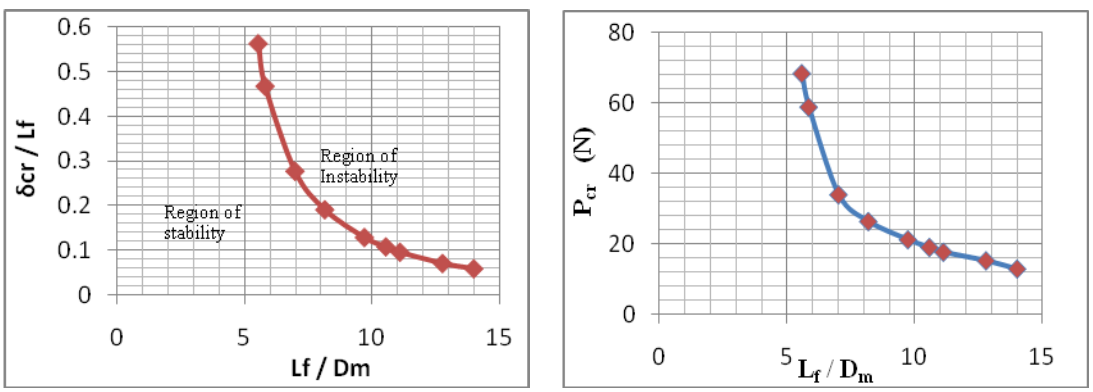
Induced shear stress is within the permissible limit. Hence the spring design is safe.

3.3. The Outcomes of Theoretical Analysis has been shown in Table 2 as Given Below.

Table 2. Straight Helical Compression Springs Made of ASTM A313 TYPE 304 (SS304)

Spring No.	(Free length / mean coil diameter) (L_f / D_m)	Helix angle at buckling. (α) Deg.	Relative Critical deflection ($\frac{\delta_{cr}}{l_f} = \xi$)	Critical load (P_{cr}) (N)	Buckling Deflection δ_{cr} (mm)	Corrected torsional shear stress (τ') N/mm^2
1	5.56	2.22	0.563	68.51	56.30	456.12
2	5.83	2.82	0.468	58.95	49.14	392.48
3	7.00	3.68	0.278	34.00	32.03	226.36
4	8.17	4.60	0.192	26.37	28.22	175.57
5	9.72	5.91	0.130	21.18	22.75	141.01
6	10.56	6.59	0.109	18.99	20.71	126.43
7	11.11	6.98	0.097	17.63	19.40	117.38
8	12.78	8.24	0.072	15.17	16.56	100.99
9	14.00	9.14	0.060	12.85	15.12	85.550

From the above analysis, it is implies that, for large value of helix angles, agreement with Haringx's results becomes poor with the experimental values. Also, significant deviations from the elementary theory occur at small number of turns or large values of helix angles, where the effective rigidities (*i.e.*, flexural, torsional and shearing rigidities) used by Haringx are inaccurate.



a) Slenderness Ratio Versus Relative Critical Deflection

b) Slenderness Ratio Versus Critical Axial Load

Figure 1. Theoretical Buckling Characteristics Curve.(for both ends fixed)

3.4 Theoretical Analysis for Conical Helical Compression Springs

The load-length characteristics of these conical springs are usually **linear** and **nonlinear**. In the non-linear phase, the spring stiffness is not constant but depends on the compression. This behavior occurs when the number of active coils decreases or increases with varying compression. The non-linear behavior of a spring can be achieved by

i) Varying the mean coil diameter in axial direction
ii) Varying the pitch.
iii) Varying the spring wire diameter along its length.

In this research findings, the non-linear behavior has been achieved by varying the mean coil diameter in axial direction and keeping the constant spacing between adjacent coils along the axis of the conic (*i.e.*, constant pitch) as shown in tale 3. The rate of the spring in the linear phase can easily be calculated but in non-linear phase, it not straight forward.

The deflection and load analysis has been carried out using the author's newly developed equation and E. Rodriguez et al [6], as follows.

Table 3. Specifications of Conical Spring Made of ASTM A313304(Stainless Steel)(Non-Telescopic)

Smallest Outer coil diameter (D_1), mm	Biggest outer coil diameter (D_2), mm	Free Length of the Coil (L_f), mm	Mean coil diameter (D_m), mm	Wire diameter (d), mm	Pitch of the coil (p), mm	Helix angle (α_o)	Spring Index C $=D_m/d$	L_f / D_m Ratio	Total Number of turns (n_o)	Active turns (n_a)
20	30	100	25.0	2	5.70	5.0	12.50	4.0	20	16
20	30	105	25.0	2	6.24	5.0	12.5	4.0	20	16
20	32	126	26.0	2	5.26	5.0	13.0	4.8	25	21
20	34	147	27.0	2	7.50	6.7	13.5	5.4	25	21
20	37	175	28.5	2	8.30	7.0	14.2	6.1	26	22
20	38	190	29.0	2	9.18	8.5	14.5	6.5	26	22
20	39	200	29.5	2	9.30	8.5	14.2	6.7	27	23
20	42	230	31.0	2	10.0	7.5	15.5	7.4	27	23
20	44	252	32.0	2	10.1	7.1	16.0	7.8	28	24

Note: All springs have squared and ground ends.
Material properties taken from hand book [5].
For ASTM A313 Type 304(SS)
E = 1.93 x10^5 N/mm^2 ; G = 70.3 x 10^3 N/mm^2 ; ρ = 79200 N/m^3. S_{ult} = 1717.5 N/ mm^2.

3.4.1 For Buckling Analysis, much of the Work Done by E. Rodriguez et al [6] along with the Authors Newly Developed Analytical Relation for Buckling has been reproduced in this Research Paper:

Initial active length of the spring, $L_a = L_f - n_i d$ (8)

Overall solid length of the spring, $L_c = L_s + n_i d$ (9)

i) For linear behavior phase

The load-length characteristic is linear since the spring rate k is constant:

$$k = \frac{Gd^4}{2n_a(D_1^2+D_2^2)(D_1+D_2)} \tag{10}$$

For $P \in [0; P_T]$: compressed length of the springs in the linear zone,

$$L = L_f - \frac{P}{k} \quad and \;\; so \quad L_T = L_f - \frac{P_T}{k} \tag{11}$$

And for $L \in [L_T; L_f]$, $P = k(L_f - L)$ (12)

ii) Nonlinear Behavior Phase

a) Compression Process Analysis**.** Along the nonlinear phase the active coils gradually stack one above the other.

$$\text{Solid length, } L_s = \left\{ max\left[0, (n_a d)^2 - \left(\frac{D_2 - D_1}{2}\right)^2\right]\right\}^{1/2} \tag{13}$$

b) At transition point *T*, the load is given by

$$P_T = \frac{Gd^4(L_a - L_s)}{8D_2^3 n_a} \tag{14}$$

Once P_T is known, length at transition is directly deduced from following equation:

$$L = L_o - P/k \quad \text{i.e.,} \quad L_T = L_o - \frac{P_T}{k} \tag{15}$$

$$\text{Load at maximum compression point,} \quad P_C = \frac{Gd^4(L_a - L_s)}{8D_1^3 n_a} \tag{16}$$

The associated length L_c can be calculated.

c) The number of current free coils) can be calculated as

$$\text{For } P \in [P_T; P_C]: \quad n_f = \frac{n_a}{D_2 - D_1}\left[\left(\frac{(L_a - L_s)Gd^4}{8Pn_a}\right)^{\frac{1}{3}} - D_1\right] \tag{17}$$

Total axial spring deflection Δ is the sum of both free coils and solid/ground coils deflections Finally

$$\Delta(P) = \frac{2PD_1^4 n_a}{Gd^4(D_2 - D_1)}\left\{\left[1 + \left(\frac{D_2}{D_1} - 1\right)\frac{n_f}{n_a}\right]^4 - 1\right\} + (L_a - L_s)\left(1 - \frac{n_f}{n_a}\right) \tag{18}$$

This equation has been derived by **E. Rodriguez *et al*** [6].

The length of a constant pitch conical spring can thus be calculated using the following formula:

$$L(P) = L_o - \left(\frac{2PD_1^4 n_a}{Gd^4(D_2 - D_1)}\left\{\left[1 + \left(\frac{D_2}{D_1} - 1\right)\frac{n_f}{n_a}\right]^4 - 1\right\} + (L_a - L_s)\left(1 - \frac{n_f}{n_a}\right)\right) \tag{19}$$

And following expressions, S. P. Timoshenko *et al* [1] ,also have been used.

d) Torsional shear stress,

$$\tau_{max} = \frac{16P_{cr}}{\pi d^3}\left[R_1 + \frac{(R_2 - R_1)\phi}{2\pi n_o}\right] \tag{20}$$

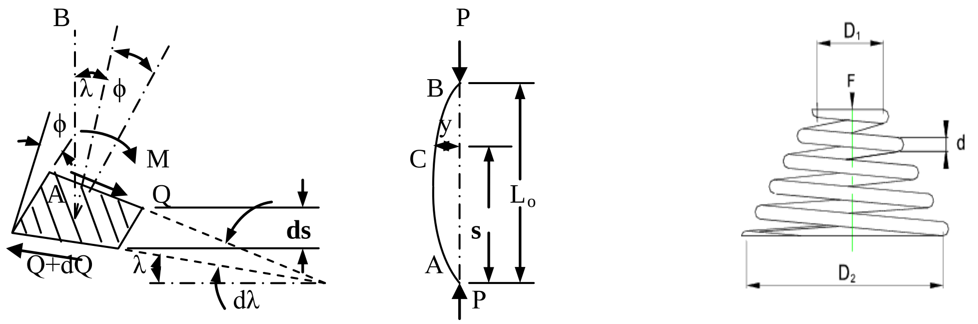
where, $\phi = 2\pi n$ and n- nth coil where the stress is being measured. It is maximum at the biggest coil in the spring.

e) Stiffness of the spring, For linear zone,

$$k_s = \frac{P}{\delta} = \frac{Gd^4}{2n'(D_1 + D_2)(D_1^2 + D_2^2)} \tag{21}$$

3.4.2 The Newly Developed Equation by Authors: For the purpose of analysis of buckling of conical helical compression spring having constant pitch, the following equation has been developed considering the effect of shearing force(Figure 4) that was previously ignored by

E.B.Wolansky[8] and perhaps this is the first such modeling for a conical spring.



(a) Shearing Deformation of Element of Spring Wire. (b) Buckling of Spring c) Conical Helical Spring

Figure 2. Buckling of Conical Spring with Shear Deformation of its Differential Element

for E/G = 2.7 $$\frac{\delta_{cr}}{l_o} = \frac{1-\sqrt{1-2.84\left(\frac{D_1}{l_o}\right)^2[ab-1.482]}}{2} \qquad (22)$$

where, $a = \left(1+\frac{D_2}{D_1}\right)$ and $b = 1+\left(\frac{D_2}{D_1}\right)^2$. For both end fixed, $l_f = \frac{l_f}{2}$

3.4.3 Accocding to Wolansky's Equation [8]:

Critical deflection,

$$\frac{\delta_{cr}}{L_f} = \frac{1}{2} - \frac{1}{2} \times \sqrt{1-\frac{C\times\pi^2\times\left(D_1^2+D_2^2\right)(1+\mu)}{L_f^2\,(2+\mu)}} \qquad (23)$$

Taking, $\mu = 0.3$ and C= 4 (for both ends are fixed)

3.4.4 Numerical Illustrations:

Example 1. For spring no.3 from table 3
D_1 =20 mm, D_2 = 32 mm , D_m =26 mm , d= 2 mm , L_f = 126 mm, p= 5.26 mm, α_o =5.5, n_a=21, n_o =25
E = 1.93 x10^5 N/mm^2 ; G = 70.3 x 10^3 N/mm^2 ; ρ = 79200 N/m^3. S_{ult} = 1717.5 N/ mm^2.

i) According newly developed equation by the author,

ii) Crtical deflection, $$\frac{\delta_{cr}}{l_o} = \frac{1-\sqrt{1-2.7892\left(\frac{2D_1}{l_o}\right)^2[ab-1.536]}}{2}$$ (for both ends fixed),

where, $a = \left(1+\frac{D_2}{D_1}\right) = 1+\frac{32}{20} = 2.6$ and $b = 1+\left(\frac{D_2}{D_1}\right)^2 = 1+\left(\frac{32}{20}\right)^2 = 3.56$

$$\frac{\delta_{cr}}{l_o}=\frac{1-\sqrt{1-2.7892\left(\frac{2\times 20}{126}\right)^2[9.256-1.536]}}{2}=0.5-1.5i$$

It is not admissible because ratio of two real quantity cannot be imaginary and hence no buckling of springs.

iii) Accocding to Wolansky's Equation [8],

Critical deflection,

$$\frac{\delta_{cr}}{L_f}=\frac{1}{2}-\frac{1}{2}\times\sqrt{1-\frac{4\times\pi^2(20^2+32^2)(1+0.3)}{126^2(2+0.3)}}=0.5-0.5i$$

It is not admissible because ratio of two real quantity cannot be imaginary and hence no buckling of springs.

Example 2. For a spring having D_1=20, D_2=45, L_f=290, n_o = 29, n_a= 25, E = 1.93 x10^5 N/mm^2 ; G = 70.3 x 10^3 N/mm^2 ; ρ = 79200 N/m^3. S_{ult} = 1717.5 N/ mm^2.

i) According to newly developed equation by the author, for both ends fixed,

$$\frac{\delta_{cr}}{l_o}=\frac{1-\sqrt{1-2.7892\left(\frac{2\times 20}{290}\right)^2[19.703-1.536]}}{2}$$

= 0.405169

Deflection at buckling, δ_{cr} = 0.405169 x 290 = 117.499mm

iii) Accocding to Wolansky's Equation [8],

$$\frac{\delta_{cr}}{L_f}=\frac{1}{2}-\frac{1}{2}\times\sqrt{1-\frac{4\times\pi^2(20^2+45^2)(1+0.3)}{290^2(2+0.3)}}=0.2025$$

Deflection at buckling = δ_{cr} = 0.2025 x 290 = 58.725 mm

Solid length, $L_s=\sqrt{(n_a d)^2-\left(\frac{D_2-D_1}{2}\right)^2}=\sqrt{(25\times 2)^2-\left(\frac{45-20}{2}\right)^2}=48.412\ mm$

Overall active length, $L_a = L_o - n_i d$ = 290 – 2 x 2 = 286 mm , n_i – factor influencing the end conditions.

L_a - L_s = 286 – 48.412 = 237.588 mm

$$P_T=\frac{Gd^4(L_a-L_s)}{8D_2^3 n_a}=\frac{70.3\times 1000\times 2^4\times 237.588}{8\times 45^3\times 25}=14.66\ N=1.494\ kg$$

For linear zone, $k_s=\frac{P}{\delta}=\frac{Gd^4}{2n'(D_1+D_2)(D_1^2+D_2^2)}$

$$=\frac{70.3\times 1000\times 2^4}{2\times 21\times (20+45)\times(20^2+45^2)}$$

= 0.1699 N/mm

Corresponding compressed length of the spring at transition point T,

$L_T = L_o - P_{k/Ks}$ (because the stiffness is constant at that point).
$L_T = 290 - (14.66 / 0.1699) = 203.714$ mm .
Therefore, $\delta_T = L_f - L_T = 290 - 203.714 = 86.286$ mm .
And according Wolansky's Equation,
Critical deflection (δ_{cr} =58.725 mm) < δ_T (= 86.286 mm)
Hence, buckling takes place in the linear zone where its stiffness will remains constant during compression.
The corresponding critical or buckling load.
$P_{cr} = \delta_{cr} \times k_s = 58.725 \times 0.1699 = 9.977\ N = 1.017\ kgf$
But according to authors equation,
Critical deflection (δ_{cr} =117.499 mm) > δ_T (= 86.286 mm) as per authors equation.
Therefore, the buckling takes place in the non- linear zone where its stiffness will not remains constant during compression.
To find the corresponding critical or buckling load, we may use Rodriguez approach[6] as elaborated below.
The no. of active coils at the transition point 'T',

$$n_f = \frac{n_a}{D_2 - D_1}\left[\left(\frac{(L_a - L_s)Gd^4}{8P_T n_a}\right)^{1/3} - D_1\right]$$

$$n_f = \left(\frac{25}{45 - 20}\right)\left[\left(\frac{(286 - 48.412) \times 70.3 \times 10^3 \times 2^4}{8 \times P_T \times 25}\right)^{1/3} - 20\right]$$

$$n_f = \frac{113.34}{15.49^{\frac{1}{3}}} - 19.23 \cong 24$$

which is true, because at transition point 'T', only biggest coil became inactive.
Also, we can check for no. of active coils at buckling stage as
Active compressed length of the spring at buckling, $L_{cr} = 286 - 117.499 = 168.5$ mm
$H = L_a - L_s = 286 - 48.412 = 237.588$ mm
Also, the active no. of coils still free to deflect at the buckling stage is given by,

$$n_f = \frac{n_a}{D_2 - D_1}\left[\left(\frac{(L_a - L_s)Gd^4}{8P_{cr} n_a}\right)^{1/3} - D_1\right]$$

$$n_f = \frac{25}{(46 - 20)}\left[\left(\frac{(286 - 48.412) \times 70.3 \times 1000 \times 2^4}{8 \times P_{cr} \times 25}\right)^{1/3} - 20\right]$$

Also, we have continuous deflection in the non-linear zone as

$$\Delta = \frac{2PD_1^4 n_a}{Gd^4(D_2 - D_1)}\left\{\left[1 + \left(\frac{D_2}{D_1} - 1\right)\frac{n_f}{n_a}\right]^4 - 1\right\} + H\left(1 - \frac{n_f}{n_a}\right).$$

Therefore, at critical stage in non-linear zone, deflection may be written as

$$\Delta = \frac{2PD_1^4 n_a}{Gd^4(D_2 - D_1)}\left\{\left[1 + \left(\frac{D_2}{D_1} - 1\right)\frac{n_f}{n_a}\right]^4 - 1\right\} + H\left(1 - \frac{n_f}{n_a}\right)$$

Now to get the corresponding critical load, we need to iterate for critical load (P_{cr}) using above two equations simultaneously until it reach to known values of n_f=24 and δ_{cr}= 117.499 mm, in the non-linear zone of compression of the conical springs.
Using **MATLAB**, as shown below.

```
>> n=29;
>> na=25;
>> G=70.3*10^3;
>> La=Lf-2*d;
>> Ls=sqrt((na*d)^2-((D2-D1)/2)^2);
>> H= La-Ls;
>> PC=G*d^4*(La- Ls)/ (8*D1^3*na);
>> PT=G*d^4*(La- Ls)/ (8*D2^3*na);
>> Ks=G*d^4/ [2*n*(D1+D2)*(D1^2 + D2^2)];
>> LT=Lf-PT/Ks;
>> P=15;
>> nf= (na/(D2-D1))* [[((La-Ls)*G*d^4)/(8*P*na)]^(1/3) - D1];
>> delta =[(2*P*na*D1^4)/((D2-D1 )*G*d^4 )*[[1+(D2/D1 -1)*(nf/na)]^4-1]+ H*(1-(nf/na)
)];
>> delta
```

First iteration

```
>> P=18;
>> nf= (na/(D2-D1))* [[((La-Ls)*G*d^4)/(8*P*na)]^(1/3) - D1];
>> delta =[(2*P*na*D1^4)/((D2-D1 )*G*d^4 )*[[1+(D2/D1 -1)*(nf/na)]^4-1]+ H*(1-(nf/na)
)];
>> delta
delta =
  122.9813
>> nf
nf =
   22.0274
```

Second iteration

```
>> P=16.5;
>> nf= (na/(D2-D1))* [[((La-Ls)*G*d^4)/(8*P*na)]^(1/3) - D1];
>> delta =[(2*P*na*D1^4)/((D2-D1 )*G*d^4 )*[[1+(D2/D1 -1)*(nf/na)]^4-1]+ H*(1-(nf/na)
)];
>> delta
delta =
  114.5926
>> nf
nf =
   23.2642
```

Third iteration

```
>> P=16.93;
```

```
>> nf= (na/(D2-D1))* [[((La-Ls)*G*d^4)/(8*P*na)]^(1/3) - D1];
>> delta =[(2*P*na*D1^4)/((D2-D1 )*G*d^4 )*[[1+(D2/D1 -1)*(nf/na)]^4-1]+ H*(1-(nf/na)
)];
>> delta
delta =
  117.1034
>> nf
nf =
  22.8947
```

Since, these above values of δ_{cr} and n_f are nearly same as what has been obtained with authors proposed equation. Hence, the corresponding buckling load from the above program is P_{cr} = 16.93 N = 1.726 kgf. and still there are 23 no. of turns active capable of axial deflection. The comparison of both method have been shown in Table 4 and theoretical analysis outcomes also shown in Table 5 and its graphical representation in Figure 3 below.

Table 4. Critical Load and Deflection of Conical Springs in Example1 and 2

Example	Method	$\xi=\delta_{cr}/L_f$	δ_{cr} (mm)	P_{cr} (N)	Comment
1	Proposed model	0.5- 1.5i	-	-	Not buckling
1	Wolansky equation	0.5-0.5 i	-	-	Not bukling
2	Proposed model	0.405169	117.499	16.93	Buckling
2	Wolansky Equation	0.2025	58.725	9.977	Buckling

Table 5. Theoretical Values of Load-Deflection of Non-telescopic Conical Springs Made of SS304

Spring No. 1		Spring No. 2		Spring No. 3		Spring No. 4		Spring No. 5		Spring No. 6		Spring no.7		Spring No. 8		Spring No. 9	
	δ	P	δ	P	δ	P	δ	P	δ	P	δ	P	δ	P	δ	P	δ
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2.2	5	2.2	5	1.5	5	1.3	5	1	5	1	5	0.9	5	0.8	5	0.7	5
4.3	10	4.3	10	3	10	4	15	3.2	15	4	20	3.7	20	3.1	20	2.7	20
6.5	15	6.5	15	6	20	6.7	25	5.3	25	7	35	6.43	35	6.2	40	5.37	40
8.7	20	8.7	20	7.6	25	9.4	30	7.5	35	10.1	45	9.2	50	9.3	60	8	60
13	23	12.9	25	10.6	27	12	35	9.7	40	13.1	50	11.9	54	12.8	70	12.6	75
18	28	15.1	28	12.2	29	17.3	40	16.9	50	17.6	58	17.3	65	15.8	77.2	16.9	85
22	32	22.6	35.3	16.4	34	30	54.7	19.1	53.3	30	74.4	30	85	30	98.9	35	115
30	38.5	35	47.4	25	43	40	64	25	62.3	45	90.6	45	102	50	123	55	140
35	43.3	45	54.9	35	52	55	79.4	40	78.7	60	106	60	115	70	140	75	160
45	51.6	50	58.9	40	56	65	88	55	94.6	75	119	75	126	85	152	95	175
50	55	55	62.1	50	67	70	91.9	70	107.8	80	123	90	137	100	162	110	184
55	58.25	60	66.3	55	71	75	95.4	85	118.3	95	132.9	95	140	115	171	120	189.5
60	60.2	65	67.8	60	75	80	98.6	95	124.2	100	135.8	100	143	125	176	135	196.5
70	63.94	73	70.8	67.3	80	85	101.6	102	127.7	114	142.8	115	151	138.6	181	147	201.3

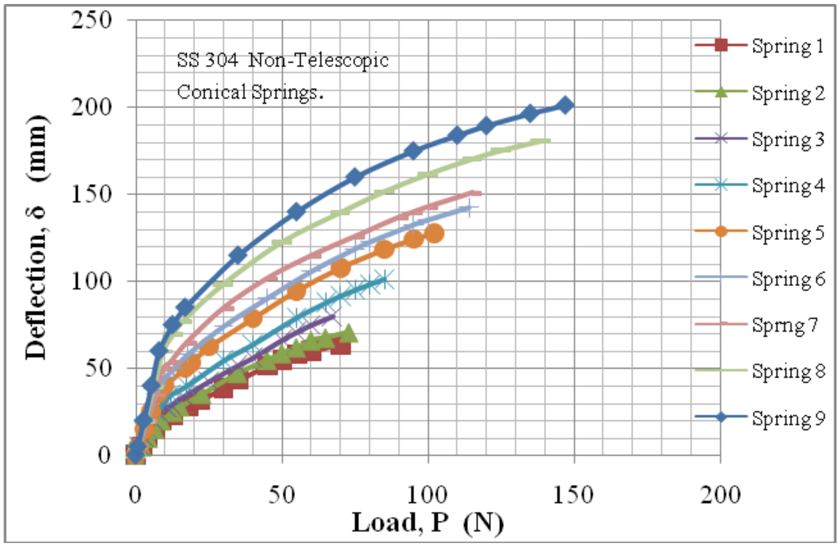


Figure 3. Theoretical Load versus Deflection Curve for Individual Springs

4 Conduct of Experiments

4.1 Test Rig

To analyze the behavior of helical compression spring, the test-rig shown in figure has been developed *and* fabricated in the institute laboratory. This spring testing machine is capable of taking load of 30 kg.

Figure 4 Spring Testing Set-Up

Test Rig Specifications: Max. height – 315 mm
Max. Diameter- 150 mm

Steps for conducting experiment:

1) Load the spring on the machine touching both the ends to the compressing plates and no load is applied on it.
2) Set the weighing machine to zero load, if necessary so that the reading on the sensor is zero.
3) Now apply the load until the sensors senses 5 mm deflection, and note down the readings on weighing machine.
4) Now apply the load until the sensors senses 5 mm deflection, and note down the readings on weighing machine.
5) Now apply the load until the sensors senses 5 mm deflection, and note down the readings on weighing machine.
6) Continue this procedure with deflection that already calculated theoretically and note down the corresponding weight on weighing machine.
7) Note down the critical deflection and corresponding load, if occurs.
8) Reverse the procedure unloading the spring through the same steps and record the corresponding load in the decrease order.

9) Repeat the same procedure with remaining springs.

4.2 Experimental Data

Testing of springs has been carried out as per the above explained procedure and the data are recorded as shown in Table 6 and 8 as given below.

Table 6. Pratical Values of Load and Deflection of Cylindrical Springs Made of ASTM313 TYPE 304 (SS304)

Spring no.1 L_f / D_m= 5.6		Spring no.2 L_f / D_m= 5.83		Spring no.3 L_f / D_m= 7.0		Spring no.4 L_f / D_m= 8.17		Spring no.5 L_f / D_m= 9.72		Spring no.6 L_f / D_m= 10.6		Spring no.7 L_f / D_m= 11.1		Spring no.8 L_f / D_m= 12.8		Spring no.9 L_f / D_m= 14	
P	δ	P	δ	P	δ	P	δ	P	δ	P	δ	P	δ	P	δ	P	δ
0 4.8 11.8 7 18.2 5 25.3 32.5 6 38.6 45 53.0 7 59.6 4 63.2 1	0 5 10 15 20 25 30 35 40 45 45. 3	0 2.06 5.88 13.4 18.7 26.8 33.1 5 39.1 4 44.5 62.5	0 5 10 15 20 25 30 35 40 44. 3	0 5.2 9.81 13.2 4 16.4 24.5 27.8 6 40.3	0 5 10 15 20 25 30 32.6	0 5.2 9 9.8 1 14. 1 18. 2 32. 1	0 5 10 15 20 24. 48	0 5.68 10.1 15.1 22.1 29.3 4 29.3 4	0 5 10 15 20 22.8 22.8	0 5 7.95 12.8 27.1 2	0 5 10 15 18 .2 9	0 4.8 10 16 23,4	0 5 10 15 18.7	0 3.13 7.9 13.8 19.2	0 5 10 15 16.4	0 4.12 5.68 16.4	0 5 10 13.3

Table 7. Comparison of Practical and Theoretical Values of Buckling Load and Deflections of the Cylindrical Springs Made of ASTM Type 304(SS304)

		Practical values			*Theoretical values*		
Spring No.	*Slenderness Ratio (L_f/D_m)*	Buckling Load P_{cr} (N)	Buckling Deflection δ_{cr} mm	*Relative Critical deflection ($\frac{\delta_{cr}}{L_f} = \xi$) (Practical)*	Buckling Load P_{cr} (N)	Buckling Deflection δ_{cr} mm	*Relative Critical deflection ($\frac{\delta_{cr}}{L_f} = \xi$) (Practical)*
1	5.56	63.21	45.34	0.4534	68.51	56.30	0.563
2	5.83	62.5	44.28	0.4217	58.95	49.14	0.468
3	**7.00**	40.4	32.59	0.2586	34.00	32.03	0.278
4	8.17	32.1	24.48	0.1665	26.37	28.22	0.192
5	9.72	29.34	22.8	0.1100	21.18	22.75	0.130
6	10.56	27.12	20.29	0.0961	18.99	20.71	0.109
7	11.11	23.4	18.7	0.0872	17.63	19.40	0.097
8	12.78	19.2	16.4	0.0651	15.17	16.56	0.072
9	14.00	16.4	13.3	0.0530	12.85	15.12	0.060

From the above Table 7 it is clear that simulation values and practical values are nearly closer to each other at almost all the points of deflection and buckling behavior of cylindrical springs has been found to be in better agreement with the theory by Haringx for both ends fixed.

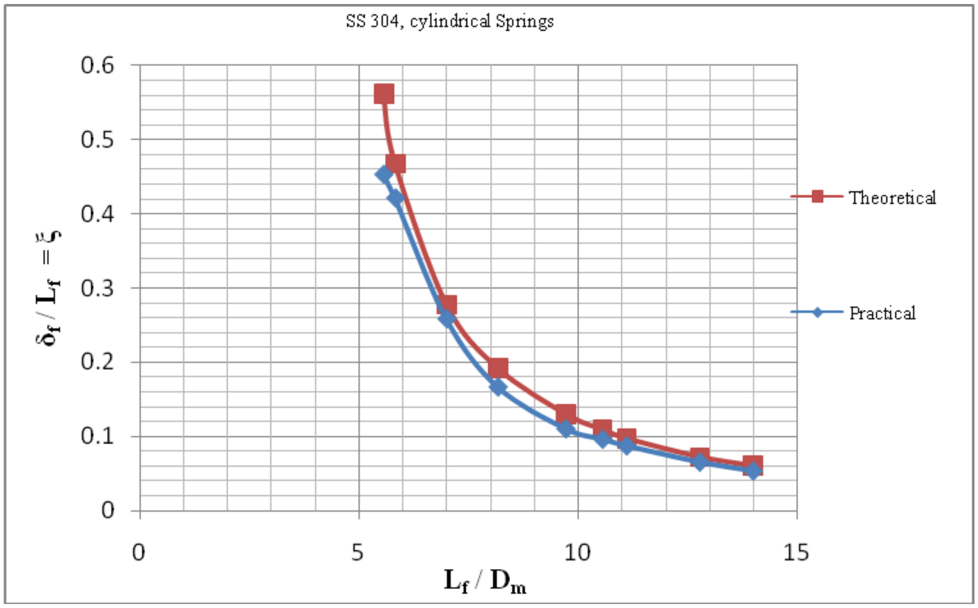


Figure 5. Comparison of Relative Critical Deflection for the same Slenderness Ratio between Theoretical and Practical Values

Table 8. Practical Values of Buckling Load and Deflections of the Non-Telescopic Conical Springs Made of ASTM A313 type 304 (SS304)

Spring no.1 L_f / D_m= 5.6		Spring no.2 L_f / D_m= 5.83		Spring no.3 L_f / D_m= 7.0		Spring no.4 L_f / D_m= 8.17		Spring no.5 L_f / D_m= 9.72		Spring no.6 L_f / D_m= 10.6		Spring no.7 L_f / D_m= 11.1		Spring no.8 L_f / D_m= 12.8		Spring no.9 L_f / D_m= 14	
P (N)	δ (mm)	P (N)	δ (mm)	P (N)	δ (mm)	P	δ (mm)	P	δ (mm)	P	δ (mm)	P	δ (mm)	P	δ (mm)	P	δ (mm)
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2.2	5	2.2	5	1.5	5	1.3	5	1	5	1	5	0.9	5	0.8	5	0.75	5
4.3	10	4.2	10	3.2	10	3.81	15	3.4	15	3.83	20	3.6	20	3.1	20	2.7	20
6.5	15	6.4	15	6.3	20	6.2	25	6.2	25	6.8	35	6.43	35	6.2	40	5.37	40
8.7	20	8.7	20	8.3	25	8.8	35	8.7	35	9.3	50	8.6	50	10.3	60	8	60
12.9	27	12.9	28	11.6	35	12	45	11.6	45	12.1	60	14	65	13.8	70	10.7	80
15.1	30	15.1	30	14.2	40	14.3	48.3	13.6	51.78	13.6	65	15.6	67.45	18.9	83	13.1	100
20.9	35.4	21.6	38.3	19.4	45.4	25	61.3	19.1	58.34	30	86.	30	88.78	30	98.3	35	133.1
30	39.1	35	48.4	25	49.1	40	76.3	25	66.8	45	101	45	108.5	50	121.9	55	137.1
35	42.8	45	54.7	35	56.8	55	86.2	40	84.2	60	114	60	119.6	70	138.4	75	158
45	48.2	50	57.7	40	61.45	65	90.4	55	98.2	75	123	75	131.3	85	152.1	95	173.2
50	50.4	55	59.2	50	66.4	70	92.9	70	109.7	80	128	90	139.6	100	161.5	110	183
55	53.3	60	62.2	55	70.7	75	95.9	85	119.7	95	134	95	143.6	115	169.7	120	188.7
60	56.8	65	65.8	60	74.8	80	98.3	95	124	100	137	100	145.7	125	175.1	135	196.1
70	63.2	76	68.7	67.3	79.8	85	101	102	128	114	142	115	150.8	138.6	181.8	147	202.5.

Note: The last readings in each are closer to the solid length and highlighted figure in each column corresponding to the point of buckling of cylindrical springs in comparison.

The above practical data has been shown in graphical format in the following Figure 6 for better understanding and insight into the behavior of springs and their buckling phenomenon. They shows that, initially all deflects linearly and after transition point the load-deflection is non-linear where the stiffness of the springs will vary. Longer the free length of the springs, more will be the linear range of deflection against the axial load.

The comparison of buckling load on the cylindrical springs against the corresponding axial load on the conical springs for the same deflection has been studied thoroughly and it has been discussed in the Section 5 as given below.

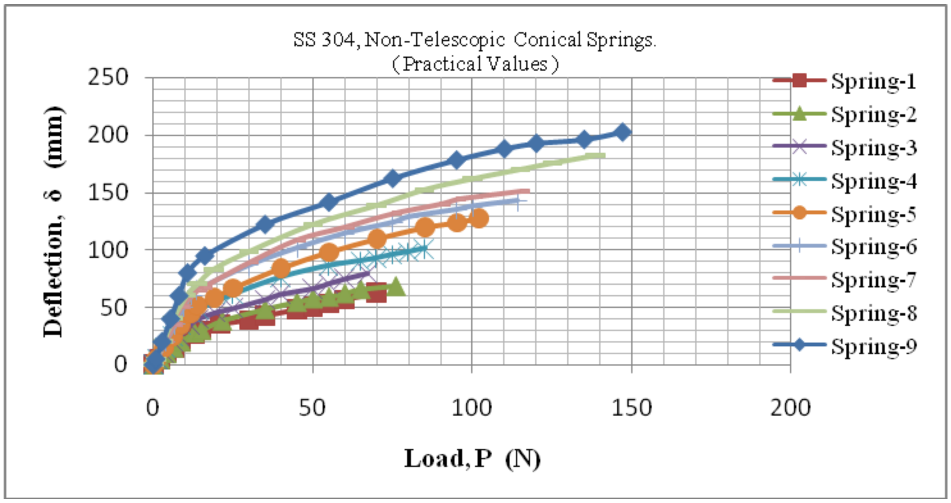


Figure 6. Buckling Load and Deflections of the Non-telescopic Conical Springs

Table 8. Comparison of Practical Buckling Loads on Cylindrical Springs Against the Corresponding Load on Conical Springs for Nearly Same Axial Deflections

Spring No.	Cylindrical helical compression springs	Axial deflection of springs	Non-telescopic conical compressions spirngs
	ASTM A313 type 304 (SS304)		ASTM A313 type 304 (SS304)
	Buckling load, P_{cr} (N)	δ_{cr}	Axial load, P (N)
1	63.21	45.34	38.6
2	62.50	44.28	29.6
3	40.40	32.59	10.9
4	32.10	24.48	6.48
5	29.34	22.80	7.30
6	27.12	20.29	4.10
7	23.40	18.70	3.30
8	19.20	16.40	2.44
9	16.40	13.30	1.58

5. Results and Discussion

From the analysis of both theoretical and practical data, it is clear that values from theoretical and practical data related to cylindrical and conical are closer to each other with difference of 2% - 4% between them. Hence the newly developed equation by the authors for conical springs gets verified. With this verification, the values of critical loads and deflections of cylindrical springs are compared against the corresponding values on the conical springs having the same free length and same deflections. For the same deflection of both cylindrical and conical springs, the percentage difference between the axial loads at the respective corresponding point of deflections is varying from 30 % to 80 % (an average of 65.5 %) as the free length of the springs increases. Thus, with the replacement of conical springs, the buckling can very well be avoided meeting all the necessary spring operations in any given systems.

6. Conclusion

After the detail analysis of behavior of both cylindrical and conical springs under the static load, it has been found that for their same deflections, the difference between the corresponding loads on them has been significant. The conical helical springs are more useful than cylindrical for greater axial deflections without buckling. However, space constraints would restrict using conical springs. This above analysis will help the designer to decide the suitability of conical springs for their replacement to avoid buckling of cylindrical springs. The newly developed equation for conical spring can be used to predict the buckling of conical spring beforehand. The conical springs are more useful where the variable stiffness is required especially in automobile systems.

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