High Order Numerical Solutionof aVolterraIntegro-Differential Equation Arising in Oscillating Magnetic Fields using Variational Iteration Method

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Abstract

In this paperwe have considered an integro-differential equation which describes the charged particle motion for certain configurations of oscillating magnetic fields. This equation contains variable coefficients with large expressions which complicate the application of any numerical method. We have used variational iteration method to find its numerical solution by developing MATHEMATICA modulae and solved a number of numerical examples. The results show high accuracy and efficiency ofour approach.

Keywords: *Variational iteration method, Integro-differential equation, Oscillating magnetic field.*

AMS Subject Classification:*34K28, 78A35, 47G20*

1. Introduction

Integro-differential equations arise in many physical phenomena of thermodynamics, chemical kinetics, fluid dynamics, mathematical biology *etc*. Due to the complexity of solving them analytically,efficient numerical techniques are required to solve them.In this paper we have considered the following integro-differential equation [1,2]:

$$\begin{cases} y''(t) + a(t)\,y(t) = g(t) + b(t)\int_0^t \cos(w_p s)\,y(s)\,ds, \\ y(0) = \alpha, \\ y'(0) = \beta. \end{cases} \qquad (1)$$

where $a(t), b(t)$ and $g(t)$ are given periodic functions of time which may be easily found in the charged particle dynamics of some field configurations. To explain the model clearly let us take three mutually orthogonal magnetic field components $B_x = B_1 \sin(\omega_p s)$, $B_x = 0$ and $B_z = B_0$.

The nonrelativistic equations of motion for a particle of mass m and charge q in third field configurations are

$$m \frac{d^2 x}{dt^2} = q \left(B_0 \frac{dy}{dt} \right), \quad (2)$$

$$m \frac{d^2 y}{dt^2} = q \left(B_1 \sin \left(\omega_p t \right) \frac{dy}{dt} - B_0 \frac{dx}{dt} \right), \quad (3)$$

$$m \frac{d^2 z}{dt^2} = q \left(-B_1 \sin \left(\omega_p t \right) \frac{dy}{dt} \right). \quad (4)$$

Integrating (2) and (4) and substituting the results into (3) yield

$$\frac{d^2 y}{dt^2} = - \left(\omega_c^2 + \omega_f^2 \sin^2 \left(\omega_p t \right) y + \omega_f \left(\sin \left(\omega_p t \right) z'(0) \right) + \omega_f^2 \omega_p \left(\sin \left(\omega_p t \right) \int_0^t \cos \left(\omega_p s \right) y(s) ds \right) \right)$$

$$+ \omega_c^2 y(0) + \omega_c x'(0),$$

where $\omega_c = q \dfrac{B_0}{m}$ and $\omega_f = q \dfrac{B_1}{m}$, which is corresponding to model(1) with the following periodic functions:

$$a(t) = \omega_c^2 + \omega_f^2 \sin^2 \left(\omega_p t \right),$$

$$b(t) = \omega_f^2 \omega_p \sin \left(\omega_p t \right),$$

$$g(t) = \omega_f \left(\sin \left(\omega_p t \right) \right) z'(0) + \omega_c^2 y(0) + \omega_c x'(0)$$

Previously, many authors have used different numerical methods to solve equation (1) e.g. homotopy analysis method [2], homotopy's perturbation method[1], Adomian method[3]. In this paper we have appliedvariational iteration method (VIM)to find solution of (1). VIM has many advantages over the other numerical methods. The calculations in applying VIM are very simple and straightforward as compared to Adomian's method since many difficulties arise while calculating Adomian polynomials. Besides, VIM requires lesser information about small parameters than as in traditional perturbation methods.

The paper is organized as follows: In Section 2, a brief description of the variation iteration method is given. In Section 3, we have applied variational iteration method to solve some numerical examples of the integro-differential equation discussed in Section 1. At last, in Section 4, conclusions are drawn.

2. Variational Iteration Method

J. H. He proposed the variational iteration method [4, 5] to solve linear and nonlinear differential equations [6-9] using an iterative scheme. He modified the general Lagrange multiplier method and constructed an iterative sequence of functions which converges to the exact solution generally. To illustrate the basic concepts of variational iteration method, we consider the following differential equation:

$$L u + N u = g(x, t)$$

where L is a linear operator and N is a non-linear operator and $g(x)$is a known real function. Now a correction functional $u(x,t)$ is constructed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left(L u_n(x,s) + N \tilde{u}_n(x,s) - g(x,s) \right) ds$$

where $\tilde{u}_n(x,t)$ is considered as a restricted variation i.e. $\delta \tilde{u}_n(x,t) = 0$. The subscript n denotes the nth order approximation. Here the optimal value ofgeneral Lagrange multiplier λ can be evaluated using the stationary conditions of the variational theory.

3. Numerical Examples

We have taken two numerical examples of (1) and we have found their solution using variational iteration method on MATHEMATICA and comparedour results with that of other methods [1, 2].

Example 1: Let us consider (1) with the following parameters:

$$\omega_p = 2,$$

$$a(t) = cos(t) \quad, \quad b(t) = sin\left(\frac{t}{2}\right),$$

$$g(t) = cos(t) - t\sin(t) + cos(t)\left(t\sin(t) + cos(t)\right) - \sin\left(\frac{t}{2}\right)\left(\frac{2}{9}\sin(3t) - \frac{t}{6}\cos(3t) + \frac{t}{2}\cos(t)\right)$$

so equations (1) becomes the following system of equation:

$$y''(t) + \cos(t)y(t) = cos(t) - t\sin(t) + cos(t)\left(t\sin(t) + cos(t)\right) - \sin\left(\frac{t}{2}\right)\left(\frac{2}{9}\sin(3t) - \frac{t}{6}\cos(3t) + \frac{t}{2}\cos(t)\right)$$

$$+ sin\left(\frac{t}{2}\right)\int_0^t \cos(2s)y(s)ds,$$

$$y(0) = 1,$$

$$y'(0) = 0,$$

To solve above system we apply variational iteration method and construct the following correctional functional:

$$y_{n+1} = y_n + \int_0^t \lambda \left(y''(\tau) + \left(a(\tau)y(\tau) \right) - g(\tau) - b(\tau) \int_0^\tau \cos(\omega s)y(s)ds \right) d\tau \qquad (5)$$

Making the above correctional functional stationary and applying restricted variations

$$\lambda''(\tau) = 0,$$

$$\lambda(\tau)_{\tau=t} = 0, \qquad (6)$$

$$1 - \lambda'(\tau)\big|_{\tau=t} = 0.$$

which yields $\lambda = \tau - t$ leading to the following iteration formula:

$$y_{n+1} = y_n + \int_0^t (\tau - t)\left(y''(\tau) + \left(a(\tau)y(\tau) \right) - g(\tau) - b(\tau) \int_0^\tau \cos(\omega s)y(s)ds \right) d\tau$$

starting from the initial condition $y(0) = 1$ we get the following approximations :

$$y_1 = \frac{14620693}{4630500} + \frac{t^2}{4} - 4\text{Cos}\left[\frac{t}{2}\right] + \frac{1}{27}\text{Cos}\left[\frac{3t}{2}\right] - \frac{1}{4}\text{Cos}[2t] + \frac{77\text{Cos}\left[\frac{5t}{2}\right]}{1125} - \frac{40\text{Cos}\left[\frac{7t}{2}\right]}{3087}$$
$$- t\text{Sin}\left[\frac{t}{2}\right] +$$

$$t\text{Sin}[t] + \text{Cos}[t](2 - t\text{Sin}[t]) + \frac{1}{9}t\text{Sin}\left[\frac{3t}{2}\right] + \frac{3}{8}t\text{Sin}[2t] + \frac{1}{75}t\text{Sin}\left[\frac{5t}{2}\right] - \frac{1}{147}t\text{Sin}\left[\frac{7t}{2}\right],$$

$$y_2 = \frac{287037449387}{300056400000} - \frac{6803t^2}{27000} - \frac{83}{54}\text{Cos}\left[\frac{t}{2}\right] - \frac{52409689\text{Cos}\left[\frac{3t}{2}\right]}{41674500} - \frac{1}{36}t^2\text{Cos}\left[\frac{3t}{2}\right]$$
$$+ \frac{18583\text{Cos}[2t]}{98784} + \frac{11315273\text{Cos}\left[\frac{5t}{2}\right]}{115762500} + \frac{1}{100}t^2\text{Cos}\left[\frac{5t}{2}\right]$$
$$- \frac{55}{648}\text{Cos}[3t] + \frac{60709\text{Cos}\left[\frac{7t}{2}\right]}{3087000} + \frac{59\text{Cos}[4t]}{432000} - \frac{13687\text{Cos}\left[\frac{9t}{2}\right]}{6001128}$$
$$+ \frac{34519\text{Cos}[5t]}{77175000} - \frac{47\text{Cos}[6t]}{889056} + \frac{133}{72}t\text{Sin}\left[\frac{t}{2}\right] - \frac{35969t\text{Sin}[t]}{176400}$$
$$+ \text{Cos}[t]\left(\frac{1732547}{661500} + \frac{t^2}{4} - t\text{Sin}[t]\right) - \frac{121t\text{Sin}\left[\frac{3t}{2}\right]}{1350}$$
$$+ \frac{2647t\text{Sin}[2t]}{4704} - \frac{1579t\text{Sin}\left[\frac{5t}{2}\right]}{110250} - \frac{29t\text{Sin}[3t]}{1296} + \frac{307t\text{Sin}\left[\frac{7t}{2}\right]}{58800} + \frac{11t\text{Sin}[4t]}{14400}$$
$$- \frac{179t\text{Sin}\left[\frac{9t}{2}\right]}{190512} + \frac{37t\text{Sin}[5t]}{367500} - \frac{t\text{Sin}[6t]}{42336}$$

We have compared our results based on this approximation with the exact solution of the problem $y = t\sin t + \cos t$. If we continue finding further approximations, the accuracy will be increasing. Even first approximation by VIM gives better results than other methods [1, 2].Theerror analysis between of variational iteration method for example 1 is shown in Table 1.

Table 1. Error Analysis of Problem 1

t	VIM(y_1)	VIM(y_2)	Exact(y)	Absolute Error $\|y - y_1\|$	Absolute Error $\|y - y_2\|$
0.1	1.004991658368365	1.00498750556099	1.0049875069427086	4.1514×10^{-6}	1.3817×10^{-9}
0.2	1.0198661422538118	1.019800356929779	1.019800444000254	6.5698×10^{-5}	8.7070×10^{-8}
0.3	1.0443191516057901	1.043991584510831	1.0439925511240078	3.2660×10^{-4}	9.6661×10^{-7}
0.4	1.0778347690554462	1.076823090632987	1.0768283309263453	1.0064×10^{-3}	5.2402×10^{-6}
0.5	1.1196745755675614	1.117276231574017	1.1172953311924743	2.3792×10^{-3}	1.9099×10^{-5}

0.6	1.1688665317041012	1.164067124195638	1.1641210989466995	4.7454×10^{-3}	5.3974×10^{-5}
0.7	1.2241969551975567	1.215666930994566	1.2157945683508722	8.4023×10^{-3}	1.2763×10^{-4}
0.8	1.2842088956649635	1.270327190330842	1.2705915820667837	1.3617×10^{-2}	2.6439×10^{-4}
0.9	1.347209670076765	1.326109990395991	1.3266041869353995	2.0605×10^{-2}	4.9419×10^{-4}
1.0	1.4112893731063234	1.380922499345068	1.3817732906760363	2.9516×10^{-2}	8.5079×10^{-4}

It can be seen from Table 1 and Figure 1 that the accuracy of second approximation y_2 of solution of problem 1 is better than that of first one and if we keep finding further approximations y_3, y_4 ... the accuracy of the solution keepson increasing.
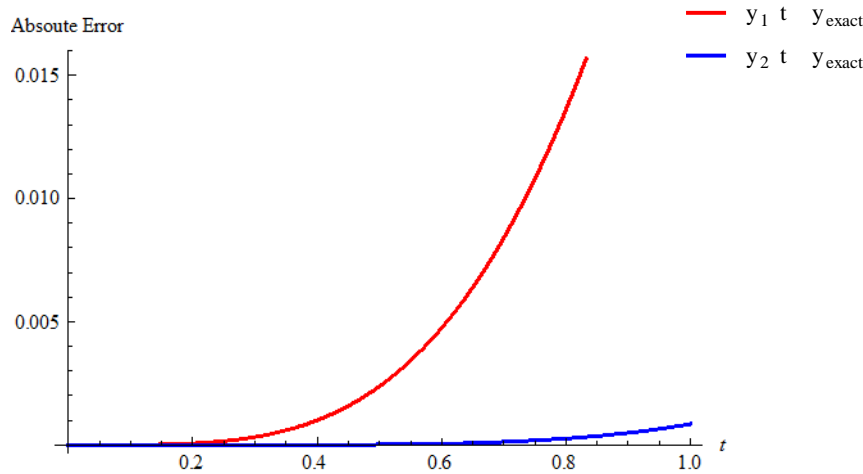

Figure 1. Comparison between Absolute Errors at First Two Approximations by VIM

Example 2: The second problem which we have considered is

$$\omega_p = 1,$$

$$a(t) = -\sin(t) \quad , b(t) = \sin(t)$$

$$g(t) = \frac{1}{9}e^{-\frac{t}{3}} - \sin(t)\left(e^{-\frac{t}{3}} + t\right) - \sin(t)\left[-\frac{3}{10}\cos(t)e^{-\frac{t}{3}} + \frac{9}{10}e^{-\frac{t}{3}}\sin(t) + \cos(t) + t\sin(t) - \frac{7}{10}\right];$$

$$\text{and } \alpha = 1, \ \beta = \frac{2}{3}.$$

On applying restricted variation on the correction functional (5) for the above parameters, we get the same stationary conditions (6), solving which we get $\lambda = \tau - t$. Now we start finding solution of this problem using (5) by taking initial solution $y_0(t) = 1 + \frac{2t}{3}$ based on

initial conditions. The exact solution of this problem is $y = e^{-\frac{t}{3}} + t$. The first approximation of this problem by VIM is evaluated as:

$$
\begin{aligned}
y_1 = 1 + \frac{2t}{3} + \frac{1}{90} \Bigg(& \frac{7121643}{27380} - \frac{549e^{-t/3}}{2} - \frac{11973t}{148} + \frac{45t^2}{2} - \frac{5t^3}{2} + 60Cos[t] \\
& - \frac{243}{5}e^{-t/3}Cos[t] + \frac{45}{4}Cos[2t] - \frac{22599e^{-t/3}Cos[2t]}{2738} - \frac{15}{4}tCos[2t] \\
& - 93Sin[t] + \frac{324}{5}e^{-t/3}Sin[t] + 30tSin[t] + \frac{15}{2}Sin[2t] \\
& - \frac{17253e^{-t/3}Sin[2t]}{2738} \Bigg)
\end{aligned}
$$

The errors at first two approximations have been displayed in Table 2.

Table 2. Absolute Errors of First Two Approximations of Problem 2 by VIM

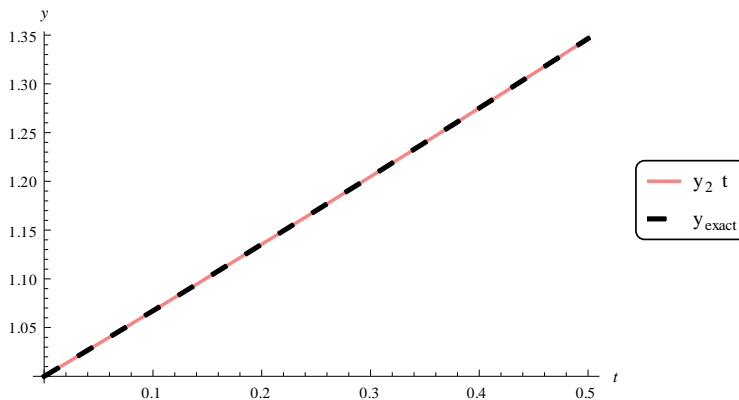| t	$|y_{exact} - y_1|$	$|y_{exact} - y_2|$
0.0	0.0	0.0
0.1	2.8163×10^{-8}	5.0900×10^{-13}
0.2	9.1175×10^{-7}	1.3237×10^{-10}
0.3	6.9879×10^{-6}	3.4433×10^{-9}
0.4	2.9646×10^{-5}	3.4759×10^{-8}
0.5	9.084×10^{-5}	2.0843×10^{-7}

Figure 2. Comparison of Second Approximation by VIM and Exact Solution of Problem 2
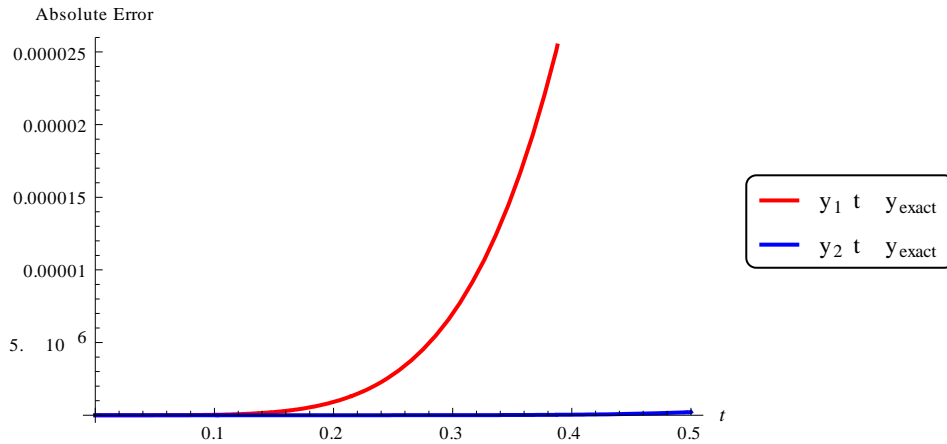
Figure 3. Comparison of Absolute Errors of First Two Approximations of Problem 2by VIM

Example 3: The Third problem which we have solved is:

$$\omega_p = 3,$$

$$a(t) = 1 \quad , b(t) = \sin(t) + \cos(t)$$

$$g(t) = -t^3 + t^2 - 11t + 4 - (\sin(t) + \cos(t))\left(-\frac{t^3}{3}\sin(3t) - \frac{t^2}{3}\cos(3t) - \frac{13}{27}\cos(3t) - \frac{13}{9}t\sin(3t) + \frac{t^2}{3}\cos(3t) \right.$$

$$\left. + \frac{16}{27}\sin(3t) + \frac{2}{9}t\cos(3t) + \frac{13}{27} \right)$$

and $\alpha = 2, \beta = -5$.

Now we find the solution of above equation by using the correctional function

$$y_{n+1} = y_n + \int_0^t \lambda \left(y''(\tau) + \left(a(\tau) y(\tau) \right) - g(\tau) - b(\tau) \int_0^\tau \cos(\omega s) y(s) ds \right) d\tau \tag{7}$$

Applying the restricted variation to above functional we get the following stationary conditions:

$$\lambda''(\tau) + \lambda(\tau) = 0,$$

$$\lambda(\tau)\big|_{\tau=t} = 0, \tag{8}$$

$$1 - \lambda'(\tau)\big|_{\tau=t} = 0$$

Solving equations (8) we get value of the multiplier $\lambda = \sin(\tau - t)$. Using this value of multiplier we get the following first approximations to the solution from (7).

$$y_1(t) = \frac{1}{101250}(-101250(-2 + t(5 + (-1 + t)t)) - 6(8804 + 625t)\text{Cos}[t]$$
$$- 625(-80 + 3t(-46 + 3t$$

$$(4 + t)))\text{Cos}[2t] + (2824 + 15t(-98 + 15t(-18 + 5t)))\text{Cos}[4t] + 6(17799$$
$$+ 625t)\text{Sin}[t] -$$

$$625(152 + 3t(-26 + 3(-6 + t)t))\text{Sin}[2t] + (544 - 15t(-358 + 15t(8 + 5t)))\text{Sin}[4t])$$

Similarly we can evaluate y_2, y_3 and further approximations. Table3 will illustrate error analysis at arbitrary points between exact solution and few approximations by VIM which indicates excellent agreement ofexact solution and VIM solutions.

Table 3. Error Comparison between Three Approximations by VIM

t	$|y - y_1|$	$|y - y_2|$	$|y - y_3|$
0.0	0.0	0.0	0.0
0.2	5.1131×10^{-6}	1.3126×10^{-10}	0.0
0.4	1.3496×10^{-4}	2.5499×10^{-8}	1.5399×10^{-12}
0.6	7.2070×10^{-4}	3.2111×10^{-7}	4.1554×10^{-11}
0.8	1.7790×10^{-3}	2.4858×10^{-7}	2.1376×10^{-10}
1.0	2.5639×10^{-3}	8.33067×10^{-6}	1.3668×10^{-9}

(a)
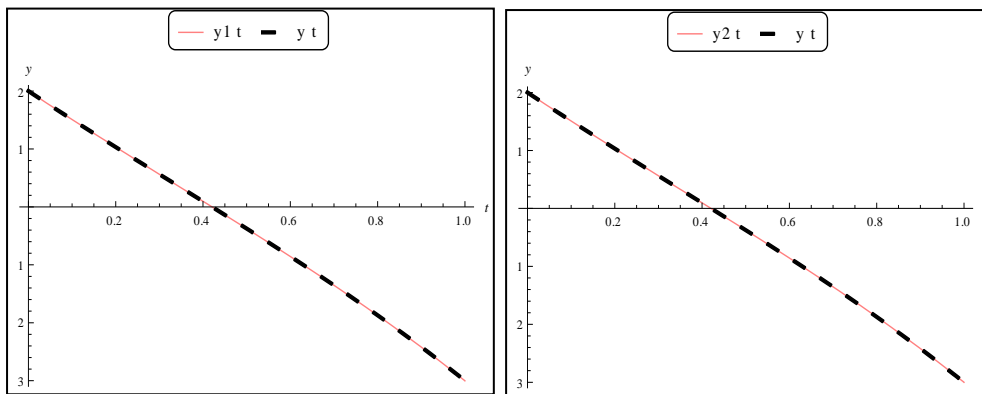
(b)
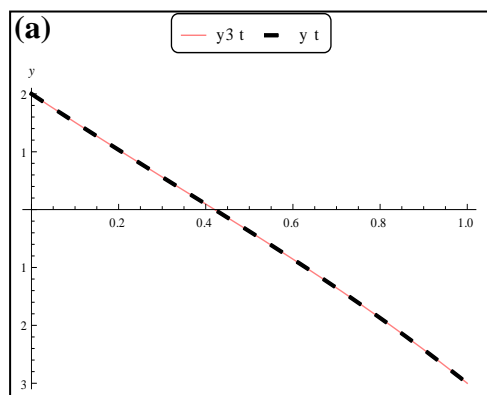
(c)

Figure 4. Comparison of the Exact Solution y(t) and (a) firstapproximation $y_1$by VIM (b) Second Approximation $y_2$by VIM (c)Third Approximation $y_3$by VIM in Problem 3

4. Conclusion

In this paper we have successfully applied variational iteration method (VIM) on an integro-differential equation with time periodic coefficients and illustrated its efficiency, wider applicability and high accuracy. We have considered three problems to solve and the computational work of this paper has been performed in MATHEMATICA software. This study shows that variational iteration method is more useful than other methods for such type of problems because the time periodic coefficients are very complex expressions and other numerical methods add more complexity to it because of their lengthy procedures. It can be easily observed from the results that as we evaluate more iterations, the numerical solution is approaching towards exact solution and the absolute error decreases drastically at each iteration.

Acknowledgement

References

[1] M. Dehghan, F. Shakeri, "Solution of an integro-differential equation arising in oscillating magnetic fields using He's homtopy perturbation method", Progress in Electro magnetics Research, vol. 78, **(2008)**, pp. 361-376.

[2] H. Saberi-Nik, S. Effati and R. Buzhabadi, "Analytic-approximate solution for an integro-differential equation arising in oscillating magnetic fields using homotopy analysis method", Iranian Journal of Optimization, vol. 3, **(2010)**, pp. 518-535.

[3] J. M. Machado and M. Tsuchida, "Solution for a class of integro-differential equations with time periodic coefficients", Appl. Math. E-Notes, vol. 2, **(2002)**, pp. 66-71.

[4] J. H. He, "Variational iteration method — a kind of non-linear analytical technique: Some examples", International Journal of Non-Linear Mechanics, vol. 34, no. 4, **(1999)**, pp. 699–708.

[5] J. He, "A new approach to nonlinear partial differential equations", Communication in Nonlinear Science & Numerical Simulation, vol. 2, Issue 4, **(1997)**, pp. 230-235.

[6] Manoj Kumar, Pratibha Joshi, Some Numerical Techniques to Solve Elliptic Interface Problems, Numerical Methods for Partial Differential Equations, 28(2012), 94-114.

[7] Pratibha Joshi, Manoj Kumar, Mathematical Model and Computer Simulation of Three Dimensional Thin Film Elliptic Interface Problems, Computers & Mathematics with Applications, 63(2012), Issue 1, Pages 25-35.

[8] M. Kumar and P. Joshi, "A Mathematical Model and Numerical Solution of a One Dimensional Steady State Heat Conduction Problem by Using High Order Immersed Interface Method on Non-Uniform Mesh", International Journal of Nonlinear Science, vol. 14, no. 1, **(2012)**, pp. 11-22.

[9] M. Kumar and P. Joshi, "Mathematical Modeling and Computer Simulation of Steady State Heat Conduction in Anisotropic Multi-Layered Bodies", International Journal of Computing Science and Mathematics, vol. 4, no. 2, **(2013)**.