Estimation of Location and Scale Parameters of Weibull Distribution Using Generalized Order Statistics under Type II Singly and Doubly Censored Data

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Abstract

In this research paper, we have discussed the estimation of location and scale parameters of Weibull distribution using concept of generalized order statistics by the best linear unbiased estimator (BLUE) and an alternative linear estimates (Gupta) from Type II singly and doubly censored samples. Tables of coefficients for best linear estimator and alternative estimator of μ and λ *have been obtained for various choices of censoring using* $n \leq 8$. *Variances and covariances of estimators of* μ and λ *based on BLUE and Gupta methods have also been presented. The computational formula and procedure used have been explained. The efficiencies of the two estimation methods have been compared under a simulation study. The findings of the study suggest that the Gupta method can be replaced by Lloyd's method which is useful when we have knowledge about the expectations and not about variance covariance matrix.*

Keywords: *Type II censoring, efficiency, best linear unbiased estimator*

1. Introduction

Ordered random variables such as order statistics and record values have gained a lot of importance during the past years. Order statistics and record values are widely used in statistical modeling and inference; both models describe random variables arranged in order of magnitude. Pawlas and Szynal [1] gave the characterization conditions for the inverse Weibull distribution and generalized extreme value distributions by moments of kth record values. Lee [2] discussed the characterization of the exponential distribution based on conditional expectation of the record values. Raqab [3] considered three parameters generalized exponential distribution and obtained the exact expressions for single and product moments of record statistics. The means, variances and covariances of the record statistics were computed for various values of the shape parameter and for some record statistics. The predictors of the future record statistics were also discussed. Abu-Youssef [4] characterized the general classes of continuous distributions by considering the conditional expectation of function of record values. Chang and Lee [5] presented the characterizations based on the identical distribution and the infinite moments of the exponential distribution by record values. Doostparast and Ahmadi [6] discussed the Bayesian estimation for the two parameters of some life distributions including: Exponential, Weibull, Pareto and Burr type XII distributions based on upper record values. Prediction, either point or interval, for future upper record values was also presented. Sultan [7] established some new recurrence relations between the single moments of record values from the modified Weibull distribution as well as between the double moments. Ahmadi and Doostparast [8] presented the Bayesian estimation as well as prediction based on k-records, when the underlying distribution has been assumed to have a general form. Baklizi [9] addressed problem of the likelihood and Bayesian estimation of the stress-strength reliability based on lower record values from the generalized exponential distribution. Ahmadi *et al.*, [10] addressed the problem of predicting future *k*-records based on *k*-record data for a large class of distributions which includes several well-known distributions such as: Exponential, Weibull (one parameter), Pareto, Burr type XII, among others. Aksop and Celebioglu [11] considered the joint distribution of any record value and an order statistics. Lee and Lim [12] discussed the characterizations of the Lomax, exponential and Pareto distributions by conditional expectations of record values. Prakash [13] studied the several distributional properties of the lower record values from the inverted exponential distribution. Afshari [14] obtained the Bayesian estimation and survival function by record values under square error and linear exponential loss functions. Further illustrations can be seen from Shawky and Abu-Zinadah [15], Soliman *et al.*, [16], Sultan [17] and Sultan *et al.*, [18]

In addition to these well-known models, several other models of ordered random variables as: sequential order statistics, order statistics with non-integral sample size and Pfeifer's record model from non-identical distributions can be constructed. These models with different interpretations can be effectively applied in reliability theory. These models of ordered random variables are described in a new form of the joint distribution of n ordered random variables as a unified approach, which in the theoretical sense, are contained in the concept of generalized order statistics given by Kamps [19]. However, very few authors have discussed estimation of the parameters of a statistical distribution based on generalized order statistics under censored samples. We have considered the estimation of location and scale parameters of Weibull distribution using concept of generalized order statistics by the best linear unbiased estimator (BLUE) and Gupta method from Type II singly and doubly censored samples.

2. Generalized Order Statistics

The Generalized Order Statistics is the most inclusive area of ordered random variables which is use broadly in everyday life .These can be used to see the distribution of first position obtained by students for many years. In this situation the marks of students will be arranged for awarding the position to the students. The distribution can be seen by taking the value of constant to be 0 *i.e.* p=0. The generalized order statistics is also valid for observing the records of some sports. One record of the sports is dependent on the other previously made record. The distribution of these records can be observed by using the Generalized Order Statistics for m=-1.

The concept of generalized order statistics is closely related to order statistics and record values.

Let $n \in N$, $p \geq 1, m_1,\ldots,m_{n-1} \in R$, $M_j = \sum_{i=j}^{n-1} m_i, 1 \leq j \leq n-1,$ be parameters such that $\gamma_j = p+n-j+M_j \geq 1$ for $j=1,2,\ldots,n-1$, $M_j = m_1+m_2+\ldots.+m_{n-1}$ and let $\tilde{m}$ $=(m_1,m_2,\ldots,m_{n-1})$, if $n\geq 2$ ($\tilde{m} \in R$ arbitrary, if $n=1$).If the random variables U(r,n,m,p) for r=1,2,…,n, possess a joint density function of the form

$$f_{u(1,n,m,p),\dots,u(n,n,m,p)}(u_1,u_2,\dots,u_n) = p\left[\prod_{j=1}^{n-1}\gamma_j\right]\left[\prod_{i=1}^{n-1}(1-u_i)^m\right](1-u_n)^{p-1} \text{ where } 0\le u_1\le\dots\le u_n\le 1$$

The random variable U(r, n, m, p) are then called uniform generalized order statistics. Generalized order statistics based on the distribution function *F* obtained from the uniform generalized order statistics using the quantile transformation is: X(r, n, m, p) =F^{-1}(U(r, n, m, p)).

Let F be the continuous distribution function then from the above definition of generalized order statistics, {X(1,n,m_1,p), X(2,n,m_2,p),…, X(n,n,m_n,p)} will denote n generalized order statistics corresponding to c.d.f. F and the p.d.f. $f(x)=\dfrac{dF(x)}{dx}$.

The joint p.d.f $f_{1,2,\dots,n}(x_1,x_2,\dots,x_n)$ of generalized order statistic is given as:

$$f_{1,2,\dots,n}(x_1,x_2,\dots,x_n) = p\prod_{j=1}^{n-1}\gamma_j\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i}[1-F(x_n)]^{p-1}f(x_i)f(x_n) \qquad (1)$$

$$F^{-1}(0)<x_1<x_2<\dots<x_n<F^{-1}(1)$$

The marginal density function of the rth Uniform Generalized Order Statistics is:

$$f_{u(r,n,m,p)}(u)=\frac{c_{r-1}}{(r-1)!}(1-u)^{\gamma_r-1}g_m{}^{r-1}(u)\ \ 0\le u\le 1$$

The marginal p.d.f $f_{r,n,m,p}$ of X(r,n,m,p) is given in

$$f_{r,n,m,p}(x)=\frac{c_{r-1}}{(r-1)!}[1-F(x)]^{\gamma_r-1}f(x)g_m{}^{r-1}[F(x)] \qquad (2)$$

$$F^{-1}(0)<x<F^{-1}(1) \text{ and } c_{r-1}=\prod_{j=1}^{r}\gamma_j$$

$$g_m(x)=\frac{1}{m+1}\left[1-(1-x)^{m+1}\right] if\ m\ne -1 \text{ and } g_m(x)=-\ln(1-x)\ if\ m=-1$$

So we can write generally

$$g_m(x)=\frac{1}{m+1}[1-(1-x)^{m+1}] \text{ for all m with } g_{-1}(x)=\lim_{m\to -1} g_m(x)$$

The joint density of the U(r, n, m, p) and U(s, n, m, p), r<s, with $m_1=m_2=\dots=m_{r-1}=\mu$, $m_{r+1}=\dots=m_{s-1}=v$ (no assumption, if s-r=1), is given by

$$f_{u(r,n,m,p),u(s,n,m,p)}(u_r,u_s)=\frac{c_{s-1}}{(r-1)!(s-r-1)!}(1-u_r)^{m_r}g_\mu{}^{r-1}(u_r)\left[h_v(\mu_s)-h_v(\mu_r)\right]^{s-r-1}(1-u_s)^{\gamma_s-1} \qquad (3)$$

$$0\le u_r\le u_s\le 1$$

$$\mathrm{h_m}(x)=\begin{cases}-\dfrac{1}{m+1}(1-x)^{m+1} & m=-1\\ \log\left(\dfrac{1}{1-x}\right) & m\ne -1\end{cases}$$

$$\mathrm{g_m}(x)=h_m(x)-h_m(0)$$

The joint density function of x(r, n, m, p) and x(s, n, m, p), r<s is given in:

$$f_{r,s,n,m,p}(x,y)=\frac{c_{s-1}}{(r-1)!(s-r-1)!}[\overline{F}(x)]^{m}[g_{m}F(x)]^{r-1}[g_{m}(F(y))-g_{m}(F(x))]^{s-r-1}[\overline{F}(y)]^{\gamma_{s}-1}f(x)f(y)$$

$$F^{-1}(0)<x<y<F^{-1}(1) \quad \text{And } \overline{F}(x)=1-F(x) \tag{4}$$

3. Results and Discussions

In this section, different results have been derived and corresponding discussions have been made for generalized order statistics from the Weibull distribution. The coefficients have been estimated under BLUE and Gupta methods under type ii singly and doubly censored samples.

3.1. Generalized Order Statistics for Weibull Distribution

In this section, the discussions have been made for generalized order statistics when p=2, m=1. The probability density function of Weibull distribution is:

$$f(y)=\frac{(y-\mu)^{\gamma-1}}{\sigma^{\gamma}}\exp\left(-\frac{(y-\mu)^{\gamma}}{\gamma\sigma^{\gamma}}\right) \qquad \mu<x<\infty,\sigma>0 \tag{5}$$

$$F(y)=1-\exp\left(-\frac{(y-\mu)^{\gamma}}{\gamma\sigma^{\gamma}}\right) \tag{6}$$

The joint p.d.f $f_{1,2,\ldots,n}(x_1,x_2,\ldots,x_n)$ of generalized order statistic is given as:

$$f_{r,n,m,p}(x)=\frac{c_{r-1}}{(r-1)!}\left[\exp\left(-\frac{(x)^{3}}{3}\right)\right]^{\gamma_{r}-1}x^{2}\exp\left(-\frac{(x)^{3}}{3}\right)\left[\frac{x^{3}}{3}\right]^{r-1} \qquad m=-1 \tag{7}$$

$$f_{r,n,m,p}(x)=\frac{c_{r-1}}{(r-1)!}\left[\exp\left(-\frac{(x)^{3}}{3}\right)\right]^{\gamma_{r}-1}x^{2}\exp\left(-\frac{(x)^{3}}{3}\right)\frac{1}{(m+1)^{r-1}}\left[1-\exp\left(-\frac{(x)^{3}}{3}\right)\right]^{r-1} \qquad m\neq-1 \tag{8}$$

where

$\gamma_r = p+(n-r)(m+1)$ *for* $p=1, m=-1$

$\gamma_r = 1$ and $C_{r-1}=1$ similarly *for* $p=1, m=0$ the $\gamma_r = 1+(n-r)$

The joint density function of x(r, n, m, p) and x(s, n, m, p), r<s for Weibull distribution is:

$$f_{r,s,n,m,p}(x,y)=\frac{c_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}\binom{r-1}{i}\sum_{i=0}^{s-r-1}\binom{s-r-1}{j}(-1)^{i}x^{3}y^{3}\left(e^{-(m+1)(s-i-1)(\frac{x^{3}}{3}}\right)\left(e^{-(m+1)(i-\gamma_{s})(\frac{y^{3}}{3}}\right) \tag{10}$$

Table 1. Means of the Generalized Order Statistics for p=2, m=1 from Weibull Distribution for 1 ≤ r ≤n ≤ 8

n\r	1	2	3	4	5	6	7	8
1	1.022207							
2	0.811326	1.233088						
3	0.708759	1.016461	1.341401					
4	0.643950	0.903185	1.129738	1.414260				
5	0.597790	0.828589	1.015078	1.206177	1.463400			
6	0.562542	0.774031	0.937706	1.092450	1.263040	1.50		
7	0.534367	0.731594	0.880122	1.014485	1.150925	1.30	1.5360	
8	0.511103	0.697210	0.834746	0.955748	1.073222	1.19	1.3446	1.56

Table 2. Variance and Covariance of Generalized order Statistics for p=2, m=1from Weibull Distribution

n\	1	2	3	4	5	6	7	8
2	0.08694							
	0.08694	0.10015						
3	0.06635							
	0.03693	0.06501						
	0.02259	0.04056	0.08253					
4	0.05477							
	0.03148	0.05069						
	0.02107	0.03430	0.05368					
	0.01252	0.02091	0.03440	0.06572				
5	0.04720							
	0.02761	0.04244						
	0.01913	0.02965	0.04219					
	0.01389	0.02164	0.03103	0.04672				
	0.00970	0.01517	0.02192	0.03334	0.06538			
6	0.04180							
	0.02471	0.03154						
	0.01746	0.02626	0.03559					
	0.01312	0.01982	0.02698	0.03683				
	0.00999	0.01514	0.02066	0.02838	0.04197			
	0.00721	0.01092	0.01496	0.02079	0.03088	0.06042		
7	0.03771							
	0.02246	0.03295						
	0.01607	0.02367	0.03115					
	0.01227	0.01825	0.02401	0.03118				
	0.00977	0.01422	0.01908	0.02473	0.03309			
	0.00753	0.01144	0.01495	0.01949	0.02632	0.03848		
	0.00564	0.00829	0.01108	0.01446	0.01982	0.03592	0.05664	

8	0.03450							
	0.02066	0.02990						
	0.01489	0.02165	0.02792					
	0.01159	0.01686	0.02174	0.02739				
	0.00918	0.01332	0.01789	0.02197	0.02806			
	0.00761	0.01144	0.01319	0.01860	0.02288	0.03031		
	0.00595	0.00836	0.01277	0.01398	0.01837	0.02471	0.03580	
	0.00455	0.00678	0.00825	0.01106	0.01396	0.01833	0.02720	0.0536

3.2. Estimation of μ *and* σ by BLUE

The estimation of the location and scale parameters μ, σ (not necessarily the mean and standard deviation) of a variate X whose distribution depends only on these two parameters. The parameters may be estimated by applying general least-squares theory to an ordered sample where the resulting estimates being unbiased, linear in the ordered observations, and of minimal variance.

Let $(X_{1,n,m,p}, X_{2,n,m,p} \ldots\ldots\ldots\ldots X_{n,n,m,p})$ be a sample of n generalized order statistics based on F(x). Let us make the transformation:

$$y_{r,n,m.p} = \frac{x_{r.n.m.p} - \mu}{\lambda} \quad r = 1, 2, \ldots, n$$

Which may be regarded as independent observations on the standardized variate $y_{r,n,m.p} = \frac{x_{r.n.m.p} - \mu}{\lambda} \qquad r = 1,2,\ldots\ldots, n$, whose distribution is parameter-free.

And $(y_{1.n.m.,p}, y_{2,n,m,p} \ldots\ldots\ldots\ldots y_{n,n,m,p})$ are then the realizations of a set of order random variables $(Y_{1.n.m.,p}, Y_{2,n,m,p} \ldots\ldots\ldots\ldots Y_{n,n,m,p})$

Let we denote

$$E(Y_i^{(n)}) = \alpha_i^{(n)} = \frac{c_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} y_i^{(n)} [1 - F(y_i^{(n)})]^{\gamma_r - 1} f(y_i^{(n)}) g_m{}^{r-1} [F(y_i^{(n)})] dx$$

$$Var(Y_i^{(n)}) = V_{ii}^{(n)} = \frac{c_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} y_i^{(n)2} [1 - F(y_i^{(n)})]^{\gamma_r - 1} f(y_i^{(n)}) g_m{}^{r-1} [F(y_i^{(n)})] dx dx - \left(\alpha_i^{(n)}\right)^2$$

$$E(Y_i^{(n)} Y_j^{(n)}) = W_{ij}^{(n)} = \frac{c_{s-1}}{(r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{y_j^{(n)}} y_i^{(n)} y_j^{(n)} [\overline{F}(y_i^{(n)})]^m [g_m F(y_i^{(n)})]^{r-1} [g_m(F(y_j^{(n)})) - g_m(F(y_i^{(n)}))]^{s-r-1} [\overline{F}(y_j^{(n)})]^{\gamma_s - 1} f(y_i^{(n)}) f(y_j^{(n)}) dx dy$$

$$Cov(Y_i^{(n)} Y_j^{(n)}) = V_{ij}^{(n)} = W_{ij}^{(n)} - \alpha_i^{(n)} \alpha_j^{(n)}$$

Where $c_{r-1} = \prod_{j=1}^{r} \gamma_j$ For $m_1 = m_2, \ldots = m_{n-1} = m$ and $\gamma_j = p + n - j + \sum_{i=j}^{n-1} m \quad ,1 < j < n-1$

$$g_m(x) = \begin{cases} -\ln(1-x) & if\ m=-1 \\ \dfrac{1}{m+1}\left[1-(1-x)^{m+1}\right] & if\ m \neq -1 \end{cases}$$

Reverting now to the original ordered observations we clearly have:

$$E(X_i^{(n)}) = \mu I + \lambda\alpha$$

where X is the vector of the X_i, α is the vector of the α_i and 1 a vector with unit elements. This equation may be written more compactly as:

$E(X_i^{(n)}) = p\theta$ where p is the (n x 2) matrix (1,α), and θ' = (μ,λ).

We also assume that the variance-covariance matrix $V^{(n)}$ is (nxn) symmetric +ve definite variance and covariance matrix whose elements are $V_{ij}^{(n)}$ and is non singular and its variance exists. The required estimator of the vector θ of parameters by using the extended principle of Least-square is given by:

$$\hat{\theta} = \begin{pmatrix} \hat{\mu} \\ \hat{\lambda} \end{pmatrix} = \left(p'\left(V^{(n)}\right)^{-1} p \right) p'\left(V^{(n)}\right)^{-1} X^{(n)}$$

Where $p'\left(V^{(n)}\right)^{-1} p = \begin{pmatrix} 1 \\ \alpha' \end{pmatrix} \left(V^{(n)}\right)^{-1} \begin{pmatrix} 1 & \alpha' \end{pmatrix} = \begin{bmatrix} 1'\left(V^{(n)}\right)^{-1} 1 & 1'\left(V^{(n)}\right)^{-1} \alpha \\ \alpha'\left(V^{(n)}\right)^{-1} 1 & \alpha'\left(V^{(n)}\right)^{-1} \alpha \end{bmatrix}$

Where $\alpha'\left(V^{(n)}\right)^{-1} 1 = 1'\left(V^{(n)}\right)^{-1} \alpha$ (being singular matrix)

If we denote by Δ the determinant of $p'\left(V^{(n)}\right)^{-1} p$, then we have

$$\left(p'\left(V^{(n)}\right)^{-1} p \right)^{-1} = \begin{bmatrix} \dfrac{1'\left(V^{(n)}\right)^{-1} 1}{\Delta} & \dfrac{1'\left(V^{(n)}\right)^{-1} \alpha}{\Delta} \\ \dfrac{\alpha'\left(V^{(n)}\right)^{-1} 1}{\Delta} & \dfrac{\alpha'\left(V^{(n)}\right)^{-1} \alpha}{\Delta} \end{bmatrix}$$

The estimates are given by

$$\dot{\mu} = \frac{1}{\Delta}\left(\alpha'\left(V^{(n)}\right)^{-1} \alpha 1' - \left(1'\left(V^{(n)}\right)^{-1} \alpha\right)\alpha' \right)\left(V^{(n)}\right)^{-1} X^{(n)} \quad (11)$$

$$\dot{\lambda} = \frac{1}{\Delta}\left(-1'\left(V^{(n)}\right)^{-1} \alpha 1' + \left(1'\left(V^{(n)}\right)^{-1} 1\right)\alpha' \right)\left(V^{(n)}\right)^{-1} X^{(n)} \quad (12)$$

$\text{var}(\theta)$ is obtained by solving

$$\lambda = \left(p'\left(V^{(n)}\right)^{-1} p\right)^{-1}$$

hence we find

$$\text{var}(\dot{\theta}) = \begin{bmatrix} \text{var}(\dot{\mu}) & \text{cov}(\dot{\mu},\dot{\lambda}) \\ \text{cov}(\dot{\mu},\dot{\lambda}) & \text{var}(\dot{\lambda}) \end{bmatrix} = \frac{\lambda^2}{\Delta}\begin{bmatrix} \alpha'\left(V^{(n)}\right)^{-1}\alpha & -1'\left(V^{(n)}\right)^{-1}\alpha \\ -1'\left(V^{(n)}\right)^{-1}\alpha & 1'\left(V^{(n)}\right)^{-1}1 \end{bmatrix} \tag{13}$$

$$i.e\ \ \text{var}(\dot{\mu}) = \frac{\lambda^2}{\Delta}\alpha'\left(V^{(n)}\right)^{-1}\alpha,\ \ \text{var}(\dot{\lambda}) = \frac{\lambda^2}{\Delta}1'\left(V^{(n)}\right)^{-1}1 \ \text{ and }\ \text{cov}(\dot{\mu},\dot{\lambda}) = -\frac{\lambda^2}{\Delta}1'\left(V^{(n)}\right)^{-1}\alpha$$

$$where\ X^{(n)} = \begin{bmatrix} x^{(n)}_{r_1+1} \\ x^{(n)}_{r_1+2} \\ . \\ . \\ x^{(n)}_{n-r_1-r_2+1} \end{bmatrix},\ \alpha' = \left[\alpha^{(n)}_{r_1+1}\quad \alpha^{(n)}_{r_1+2}\quad .\quad .\quad \alpha^{(n)}_{n-r_1-r_2+1}\right] \text{ and } V = \begin{bmatrix} v^{(n)}_{r_1+1,r_1+1} & v^{(n)}_{r_1+1,r_1+2} & \cdots & v^{(n)}_{r_1+1,n-r_1-r_2} \\ v^{(n)}_{r_1+1,r_1+2} & v^{(n)}_{r_1+2,r_1+2} & \cdots & v^{(n)}_{r_1+1,n-r_1-r_2} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ v^{(n)}_{n-r_1-r_2+1,r_1+1} & . & \cdots & v^{(n)}_{n-r_1-r_2+1,n-r_1-r_2+1} \end{bmatrix}$$

These estimates μ and λ are unbiased linear in the observations and have minimum variance among the class of all linear estimates.

The coefficients for BLUE of the location and scale parameter based on generalized order statistics values of Weibull distribution from singly and doubly censored samples have been obtained but not presented here.

Table 3. Variances and Covariances of BLUEs for Location and Scale based on the GOS from the Weibull Distribution from Complete Samples. Each Value may be Multiplied by λ^2

n	r_1	r_2	$\text{Var}(\hat{\mu})$ Lloyd	$\text{Var}(\hat{\lambda})$ Lloyd	$\text{cov}(\dot{\mu},\dot{\lambda})$	n	r_1	r_2	$\text{Var}(\hat{\mu})$ Lloyd	$\text{Var}(\hat{\lambda})$ Lloyd	$\text{cov}(\dot{\mu},\dot{\lambda})$
3	0	1	0.507051	0.607442	-0.526158		1	4	0.530820	0.759486	-0.618079
	1	0	0.867595	0.628823	-0.714381		2	0	0.200708	0.147325	-0.161741
4	0	1	0.258581	0.280751	-0.249616		2	1	0.265649	0.215873	-0.228462
	0	2	0.749751	0.696777	-0.701656		2	2	0.385016	0.355552	-0.357586
	1	0	0.388595	0.283907	-0.315504		2	3	0.738525	0.792411	-0.750566
	1	1	1.004365	0.624944	-0.773767		3	0	0.314364	0.211218	-0.246958
	2	0	0.432643	0.632267	-0.496973		3	1	0.478131	0.355114	-0.400468
5	0	1	0.172858	0.177779	-0.160171		3	2	0.945535	0.795268	-0.854043
	0	2	0.229860	0.292454	-0.241017		4	0	0.550847	0.332930	-0.416612
	0	3	0.379669	0.646329	-0.471263		4	1	1.150644	0.768701	-0.927860

	1	0	0.244257	0.187578	-0.200249		5	0	0.833023	0.447291	-0.596251
	1	1	0.354568	0.321742	-0.321904		0	1	0.085197	0.081196	-0.074305
	1	2	0.656380	0.728628	-0.672337		0	2	0.095943	0.102514	-0.112901
	2	0	0.457453	0.311597	-0.362855		0	3	0.111807	0.137210	-0.112901
	2	1	0.918558	0.735416	-0.804924		0	4	0.133861	0.192644	-0.147871
	3	0	1.171017	0.686502	-0.880077		0	5	0.174913	0.308462	-0.216819
	0	1	0.128008	0.127918	-0.115774		0	6	0.284674	0.666544	-0.415070
	0	2	0.156009	0.184117	-0.155443		1	0	0.109429	0.086363	-0.088794
	0	3	0.205439	0.298398	-0.230602		1	1	0.129494	0.113475	-0.112118
	0	4	0.301896	0.534664	-0.381564		1	2	0.152033	0.148415	-0.140180
	1	0	0.150114	0.121457	-0.123855		1	3	0.196679	0.219894	-0.196672
	1	1	0.188768	0.175622	-0.169612		1	4	0.270454	0.351895	-0.295355
6	1	2	0.253957	0.277720	-0.251194		1	5	0.486644	0.767587	-0.595135
	1	3	0.408239	0.545383	-0.454407		2	0	0.150695	0.119280	-0.120734
	2	0	0.282042	0.199335	-0.225216	8	2	1	0.178112	0.144768	-0.151123
	2	1	0.422569	0.341668	-0.366644		2	2	0.260475	0.238100	-0.238799
	2	2	0.817147	0.770293	-0.777893		2	3	0.345381	0.355134	-0.338483
	3	0	0.508823	0.321955	-0.391974		2	4	0.676726	0.808665	-0.726137
	3	1	1.048386	0.756971	-0.876451		3	0	0.218220	0.149186	-0.171457
	4	0	1.279710	0.702890	-0.933904		3	1	0.302751	0.229102	-0.253647
	0	1	0.103049	0.100087	-0.091262		3	2	0.417221	0.350482	-0.371522
	0	2	0.119612	0.133316	-0.114666		3	3	0.877393	0.833991	-0.843219
	0	3	0.144885	0.189918	-0.152541		4	0	0.337888	0.212486	-0.258491
	0	4	0.189386	0.304590	-0.223976		4	1	0.527502	0.366559	-0.429413
7	0	5	0.309367	0.661839	-0.431011		4	2	1.057914	0.816358	-0.917858
	1	0	0.099977	0.108752	-0.111962		5	0	0.599773	0.341152	-0.442055
	1	1	0.164498	0.145286	-0.143964		5	1	1.227150	0.770994	-0.961356
	1	2	0.211333	0.213308	-0.200406		6	0	1.465213	0.732101	-1.023728
	1	3	0.292700	0.344743	-0.303821						

3.3. Estimation of μ *and* σ by Gupta (1952) Method

Gupta (1952) has proposed a very simple method applicable when the variance covariance matrices are unknown and we have knowledge about the expectations only. The coefficients of these linear estimates are obtained by assuming the variance matrix to be a unit matrix. Let the linear estimates be:

$$\dot{\mu} = \sum_{i=1+r_1}^{n-r_2} b_i x_{(i)} \text{ and } \dot{\lambda} = \sum_{i=1+r_1}^{n-r_2} c_i x_{(i)} \quad \text{Where } b_i = \frac{1}{n-r_1-r_2} - \frac{(\mu_i - \bar{\mu}_k)}{\sum_{i=r_1+1}^{n-r_2} (\mu_i - \bar{\mu}_k)^2}, \quad c_i = \frac{(\mu_i - \bar{\mu}_k)}{\sum_{i=r_1+1}^{n-r_2} (\mu_i - \bar{\mu}_k)^2} \text{ and}$$

$$\bar{\mu}_k = \frac{1}{n-r_1-r_2} \sum_{j=r_1+1}^{n-r_2} \mu_j$$

In matrix notation the variance of the estimates can also be written as

$\mathbf{V}(\overset{*}{\boldsymbol{\mu}}) = \boldsymbol{\lambda}^2\, \mathbf{b}'\mathbf{V}\mathbf{b}$ and $\mathbf{V}(\overset{*}{\boldsymbol{\lambda}}) = \boldsymbol{\lambda}^2\, \mathbf{c}'\mathbf{V}\mathbf{c}$

The coefficients for Gupta method of the location and scale parameter based on generalized order statistics values of Weibull distribution from singly and doubly censored samples have been obtained but not presented here.

Table 4. Variances and Covariances of BLUEs for Location and Scale based on the GOS from the Weibull Distribution from Singly and Doubly Censored Samples. Each value may be multiplied by λ^2

n	r_1	r_2	$Var(\hat{\mu})$ Gupta	$Var(\hat{\lambda})$ Gupta	n	r_1	r_2	$Var(\hat{\mu})$ Gupta	$Var(\hat{\lambda})$ Gupta
3	0	1	0.507051	0.607442		1	4	0.530820	0.759487
	1	0	0.867887	0.629015		2	0	0.223320	0.163080
4	0	1	0.255294	0.281603		2	1	0.272638	0.209583
	0	2	0.749750	0.696777		2	2	0.385600	0.367847
	1	0	0.397341	0.287879		2	3	0.738524	0.792411
	1	1	1.004363	0.624944		3	0	0.343431	0.229925
	2	0	0.432643	0.632268		3	1	0.483715	0.350842
5	0	1	0.174142	0.179748		3	2	0.945536	0.795268
	0	2	0.230554	0.295563		4	0	0.551408	0.334473
	0	3	0.379668	0.646330		4	1	1.151408	0.768700
	1	0	0.254836	0.193704		5	0	0.833023	0.447291
	1	1	0.355517	0.321913	8	0	1	0.087969	0.084045
	1	2	0.656381	0.728627		0	2	0.107435	0.098926
	2	0	0.477030	0.322406		0	3	0.114125	0.142799
	2	1	0.918559	0.735415		0	4	0.136012	0.199680
	3	0	1.171019	0.686503		0	5	0.176644	0.316053
6	0	1	0.128866	0.129513		0	6	0.284674	0.666544
	0	2	0.175708	0.188309		1	0	0.117352	0.090937
	0	3	0.439783	0.304422		1	1	0.133893	0.116707
	0	4	0.301337	0.533272		1	2	0.159410	0.156751
	1	0	0.174515	0.135778		1	3	0.198306	0.241645
	1	1	0.206430	0.187080		1	4	0.270754	0.353549
	1	2	0.263074	0.283061		1	5	0.485598	0.767587
	1	3	0.409680	0.547228		2	0	0.166562	0.120055
	2	0	0.300221	0.209376		2	1	0.194557	0.157097
	2	1	0.425852	0.343039		2	2	0.261705	0.238425
	2	2	0.817148	0.770296		2	3	0.345477	0.355151
	3	0	0.537234	0.336857		2	4	0.676727	0.808664
	3	1	1.048384	0.756971		3	0	0.241868	0.162017
	4	0	1.279789	0.702892		3	1	0.312315	0.234086
7	0	1	0.105395	0.102534		3	2	0.418343	0.350866
	0	2	0.133158	0.137473		3	3	0.877394	0.833992

0	3	0.146746	0.195525	4	0	0.371558	0.229702
0	4	0.190925	0.311037	4	1	0.534379	0.369652
0	5	0.309367	0.661840	4	2	1.057914	0.816357
1	0	0.118098	0.148869	5	0	0.646816	0.363616
1	1	0.169831	0.149326	5	1	1.227149	0.770999
1	2	0.213775	0.216037	6	0	1.465215	0.732103
1	3	0.292809	0.345719				

Table 5. Relative Efficiency of BLUE to Alternative Linear Estimate for Location Parameter based on Generalized Order Statistics from Singly and Doubly Censored Samples for3≤n≤8

n	r_1	r_2	$Var(\hat{\mu})$ Lloyd	$Var(\hat{\mu})$ Gupta	Efficiency (%)	$Var(\hat{\lambda})$ Lloyd	$Var(\hat{\lambda})$ Gupta	Efficiency (%)
3	0	1	0.507051	0.507051	1.000000	0.607442	0.607442	1.000000
	1	0	0.867595	0.867887	0.999664	0.628823	0.629015	0.999695
4	0	1	0.258581	0.255294	1.012875	0.280751	0.281603	0.996974
	0	2	0.749751	0.749750	1.000001	0.696777	0.696777	1.000000
	1	0	0.388595	0.397341	0.977989	0.283907	0.287879	0.986203
	1	1	1.004365	1.004363	1.000002	0.624944	0.624944	1.000000
	2	0	0.432643	0.432643	1.000000	0.632267	0.632268	0.999998
5	0	1	0.172858	0.174142	0.992627	0.177779	0.179748	0.989046
	0	2	0.229860	0.230554	0.99699	0.292454	0.295563	0.989481
	0	3	0.379669	0.379668	1.000003	0.646329	0.646330	0.999998
	1	0	0.244257	0.254836	0.958487	0.187578	0.193704	0.968374
	1	1	0.354568	0.355517	0.997331	0.321742	0.321913	0.999469
	1	2	0.656380	0.656381	0.999998	0.728628	0.728627	1.000001
	2	0	0.457453	0.477030	0.958961	0.311597	0.322406	0.966474
	2	1	0.918558	0.918559	0.999999	0.735416	0.735415	1.000001
	3	0	1.171017	1.171019	0.999998	0.686502	0.686503	0.999999
6	0	1	0.128008	0.128866	0.993342	0.127918	0.129513	0.987685
	0	2	0.156009	0.175708	0.887888	0.184117	0.188309	0.977739
	0	3	0.205439	0.208417	0.985711	0.298398	0.304422	0.980212
	0	4	0.301896	0.301337	1.001855	0.534664	0.533272	1.00261
	1	0	0.150114	0.174515	0.860178	0.121457	0.135778	0.894526
	1	1	0.188768	0.206430	0.914441	0.175622	0.187080	0.938753
	1	2	0.253957	0.263074	0.965344	0.277720	0.283061	0.981131
	1	3	0.408239	0.409680	0.996483	0.545383	0.547228	0.996628
	2	0	0.282042	0.300221	0.939448	0.199335	0.209376	0.952043
	2	1	0.422569	0.425852	0.992291	0.341668	0.343039	0.996003
	2	2	0.817147	0.817148	0.999999	0.770293	0.770296	0.999996
	3	0	0.508823	0.537234	0.947116	0.321955	0.336857	0.955762
	3	1	1.048386	1.048384	1.000002	0.756971	0.756971	1.000000

	4	0	1.279710	1.279789	0.999938	0.702890	0.702892	0.999997
	0	1	0.103049	0.105395	0.977741	0.100087	0.102534	0.976135
	0	2	0.119612	0.133158	0.898271	0.133316	0.137473	0.969761
	0	3	0.144885	0.146746	0.987318	0.189918	0.195525	0.971323
	0	4	0.189386	0.190925	0.991939	0.304590	0.311037	0.979273
	0	5	0.309367	0.309367	1.000000	0.661839	0.661840	0.999998
	1	0	0.099977	0.118098	0.84656	0.108752	0.148869	0.730521
	1	1	0.164498	0.169831	0.968598	0.145286	0.149326	0.972945
	1	2	0.211333	0.213775	0.988577	0.213308	0.216037	0.987368
	1	3	0.292700	0.292809	0.999628	0.344743	0.345719	0.997177
7	1	4	0.530820	0.530820	1.000000	0.759486	0.759487	0.999999
	2	0	0.200708	0.223320	0.898746	0.147325	0.163080	0.903391
	2	1	0.265649	0.272638	0.974365	0.215873	0.209583	1.030012
	2	2	0.385016	0.385600	0.998485	0.355552	0.367847	0.966576
	2	3	0.738525	0.738524	1.000001	0.792411	0.792411	1.000000
	3	0	0.314364	0.343431	0.915363	0.211218	0.229925	0.918639
	3	1	0.478131	0.483715	0.988456	0.355114	0.350842	1.012176
	3	2	0.945535	0.945536	0.999999	0.795268	0.795268	1.000000
	4	0	0.550847	0.551408	0.998983	0.332930	0.334473	0.995387
	4	1	1.150644	1.151408	0.999336	0.768701	0.768700	1.000001
	5	0	0.833023	0.833023	1.000000	0.447291	0.447291	1.000000
	0	1	0.085197	0.087969	0.968489	0.081196	0.084045	0.966101
	0	2	0.095943	0.107435	0.893033	0.102514	0.098926	1.03627
	0	3	0.111807	0.114125	0.979689	0.137210	0.142799	0.960861
	0	4	0.133861	0.136012	0.984185	0.192644	0.199680	0.964764
	0	5	0.174913	0.176644	0.990201	0.308462	0.316053	0.975982
	0	6	0.284674	0.284674	1.000000	0.666544	0.666544	1.000000
	1	0	0.109429	0.117352	0.932485	0.086363	0.090937	0.949701
	1	1	0.129494	0.133893	0.967145	0.113475	0.116707	0.972307
	1	2	0.152033	0.159410	0.953723	0.148415	0.156751	0.94682
	1	3	0.196679	0.198306	0.991796	0.219894	0.241645	0.909988
8	1	4	0.270454	0.270754	0.998892	0.351895	0.353549	0.995320
	1	5	0.486644	0.485598	1.002154	0.767587	0.767587	1.000000
	2	0	0.150695	0.166562	0.904738	0.119280	0.120055	0.993545
	2	1	0.178112	0.194557	0.915475	0.144768	0.157097	0.92152
	2	2	0.260475	0.261705	0.9953	0.238100	0.238425	0.998637
	2	3	0.345381	0.345477	0.999722	0.355134	0.355151	0.999952
	2	4	0.676726	0.676727	0.999999	0.808665	0.808664	1.000001
	3	0	0.218220	0.241868	0.902228	0.149186	0.162017	0.920805
	3	1	0.302751	0.312315	0.969377	0.229102	0.234086	0.978709
	3	2	0.417221	0.418343	0.997318	0.350482	0.350866	0.998906
	3	3	0.877393	0.877394	0.999999	0.833991	0.833992	0.999999
	4	0	0.337888	0.371558	0.909382	0.212486	0.229702	0.925051

4	1	0.527502	0.534379	0.987131	0.366559	0.369652	0.991633
4	2	1.057914	1.057914	1.000000	0.816358	0.816357	1.000001
5	0	0.599773	0.646816	0.92727	0.341152	0.363616	0.938221
5	1	1.227150	1.227149	1.000001	0.770994	0.770999	0.999994
6	0	1.465213	1.465215	0.999999	0.732101	0.732103	0.999997

4. Conclusions

The performance of the different estimation methods can be compared in terms of relative efficiency which is defined as the ratio of variances of the estimator. Here the relative efficiencies have been calculated for $n \le 8$ under singly and doubly censored samples for all three cases of generalized order statistics, order statistics and upper record values. From the tables, showing relative efficiencies, it can be seen that the relative efficiency of Lloyd's method to the Gupta method is very high in all cases. So we can replace Gupta method by Lloyd's method which is useful when we have knowledge about the expectations and not about variance covariance matrix.

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