Bayesian Analysis of the Kumaraswamy Distribution under Failure Censoring Sampling Scheme

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Abstract

This study seeks to focus on Bayesian and non-Bayesian estimation for the shape parameter of the Kumaraswamy distribution under type-II censored samples. Maximum likelihood estimation and Bayes estimation have been obtained using asymmetric loss functions. Posterior predictive distributions along with posterior predictive intervals have been derived under simple and mixture priors. Elicitation of hyper-parameter through prior predictive approach has also been discussed. As analytical comparison is difficult, so comparisons among these estimators have been made using Monte Carlo simulation study and some interesting comparisons have been presented. The findings of the study indicate that the Bayes estimation is superior to classical estimation under the suitable prior.

Keywords: *Censored sampling, prior elicitation, posterior risk, loss Functions*

1. Introduction

Kumaraswamy [1] proposed a two-parameter Kumaraswamy distribution on (0, 1), and denoted by Kum (α, β). Its cumulative distribution function is given by

$$F(x;\alpha,\beta)=1-(1-x^{\alpha})^{\beta},\ x\in(0,1), \tag{1}$$

where $\alpha>0$ and $\beta>0$ are the shape parameters. Equation (1) compares extremely favorable in terms of simplicity with the Beta cdf which is given by the incomplete beta function ratio. It has many of the same properties as the beta distribution but has some advantages in terms of tractability. Its density is:

$$f(x;\alpha,\beta)=\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1},x\in(0,1). \tag{2}$$

This distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as the height of individuals, scores obtained on a test, atmospheric temperatures, hydrological data such as daily rain fall, daily stream flow, *etc.* Kumaraswamy [1] and Ponnambalam, *et al.*, [2] have pointed out that depending on the choice of the parameter α and β Kumaraswamy's distribution can be used to approximate many distributions, such as uniform, triangular, or almost any single model distribution and can also reproduce results of beta distribution. Nadarajah [3] has discussed that the Kumaraswamy distribution is a special case of the three parameter beta distribution. The basic properties of the distribution have been given by Jones [4]. Garg [5] considered the generalized order statistics from Kumaraswamy distribution.

Censoring is of supreme importance in reliability studies. It has many types each of whom can be used in analysis of different kinds of data representing various real life circumstances. The situation may arise when complete information regarding all the units in the sample cannot be obtained. For example, the measuring instrument to be used, may not be capable of measuring the items above or below a particular point or the measurement of units above or below a certain point may not be of interest. For illustration, suppose it is desired to estimate the average life of electric bulb produced in a certain factory. The simple method would be to take a certain number of bulbs at random and burn them out to get the required number of bulbs for analysis. Instead of wasting the bulbs it might be decided to stop the experiment when a fixed number have burnt out. The random sample hence obtained would be a censored sample Type II. Further, the biologists are often required to perform experiments on animals (say, rabbits or mice) to determine the effect of certain drugs on them. A fixed number of animals are exposed to the drug for this purpose and their reaction times are observed. Experience shows that some animals take an extremely long time to react. If instead of waiting until all animals have reacted, the experiment is stopped when a fixed number have reacted it will be called type ii censoring, which may result in economical experimentation. Wingo [6] has considered Maximum likelihood estimation of Burr XII distribution under type II censoring. Howlader and Hossian [7] investigate the Bayesian estimation and prediction for Rayleigh based on type II censored data. Singh, *et al.*, [8] have discussed estimation of the parameter for exponentiated -Weibull family under type-II censoring scheme. Parakash [9] studied the Bayesian shrinkage approach in Weibull under Type II censored data. Gholizadeh, *et al.*, [10] have studied the Kumaraswamy distribution under progressively type II censored data.

The Kumaraswamy distribution does not seem to be very familiar to the statisticians and has not been investigated in much detail under the Bayesian paradigm. The purpose of this study is to obtain the estimates for the parameter assuming different asymmetric loss functions. Our main object is to study the classical and Bayes estimation procedures for the shape parameter of the Kumaraswamy distribution based on type II censored sample. The results obtained in this paper can be generalized to the estimation of the Kumaraswamy distribution based on complete sample.

The layout of the paper is as follow. In Section 2, classical estimation of the parameter β based on type II censored sample has been discussed. Loss function and the Bayesian estimator under informative priors have been derived in Section 3 and 4 respectively. Method of Elicitation of the hyper-parameters via prior predictive approach has been discussed in Section 5. Posterior predictive distribution and posterior predictive intervals have been derived in Section 6. Simulation study has been performed in Section 7. Some concluding remarks have been given in the last section.

2. Classical Estimation

This section covers the classical estimation of the shape parameter of the Kumaraswamy distribution. The estimators along with their mean square errors (MSEs) have been derived under maximum likelihood estimation (MLE) and uniformly minimum variance unbiased estimation (UMVUE).

2.1. Maximum Likelihood Estimation

In the failure censoring scheme, the n experimental units are placed under observation in a typical life test and the number of uncensored observations r is predetermined. The data (collected) consist of observations $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(r)}$ are the ordered lifetimes of these life testing items, this means that we have no information about survival item $(n-r)$ except that their lifetimes are greater than $x_{(r)}$. The experiment is terminated when the r^{th} item fails and remaining $(n-r)$ items are regarded as censored data. The likelihood function for $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(r)}$ failed observations, as given by Cohen [11], is:

$$L(\alpha, \beta|\mathbf{x}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f\left(x_{(i)};\alpha,\beta\right)\left[1-F\left(x_{(r)};\alpha,\beta\right)\right]^{n-r}. \quad (3)$$

Where

$f(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}$, $\alpha,\beta > 0$, $x \in (0,1)$. and
$F(x;\alpha,\beta) = 1-(1-x^{\alpha})^{\beta}$, $x \in (0,1)$.

$$L(\alpha, \theta|\mathbf{x}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}(x_{(i)})^{\alpha-1}\prod_{i=1}^{r}(1-x_{(i)}^{\alpha})^{\beta-1}\left(1-x_{(r)}^{\alpha}\right)^{\beta(n-r)}. \quad (4)$$

After some algebra, we obtain the maximum likelihood estimate (MLE) $\hat{\beta}_{MLE}$ as following

$$\hat{\beta}_{MLE} = \frac{r}{-\left\{\left(\sum_{i=1}^{r}\ln\left(1-x_{(i)}^{\alpha}\right)+(n-r)\ln\left(1-x_{(r)}^{\alpha}\right)\right\}} = \frac{r}{\varphi},$$

where $\varphi = -\left\{\left(\sum_{i=1}^{r}\ln\left(1-x_{(i)}^{\alpha}\right)+(n-r)\ln\left(1-x_{(r)}^{\alpha}\right)\right\}$.

The MSE for MLE of $\hat{\beta}$. is: $\quad \text{MSE}(\hat{\beta}_{MLE}) = \frac{(r+2)\beta^2}{(r-1)(r-2)}$

Here, we obtain the uniformly minimum variance unbiased estimator (UMVUE) of $\hat{\beta}$. Since family of density (2) belongs to an exponential family, therefore statistic φ is a complete sufficient statistic for $\hat{\beta}$. For detail see Gupta and Kundu [12]. Hence, the UMVUE of $\hat{\beta}$ is:

$$\hat{\beta}_{UMVUE} = \frac{r-1}{-\left\{\left(\sum_{i=1}^{r}\ln\left(1-x_{(i)}^{\alpha}\right)+(n-r)\ln\left(1-x_{(r)}^{\alpha}\right)\right\}} = \frac{r-1}{\varphi}.$$

And

$$\text{MSE}(\hat{\beta}_{UMVUE}) = \frac{\beta^2}{(r-2)}.$$

3. Loss Function

A loss function represents losses incurred when we estimate the parameter β by $\hat{\beta}$. A number of asymmetric loss functions are proposed for use, among these, we use the following three loss functions.

3.1. Degroot Loss Function (DLF)

DeGroot [13] discussed different types of loss functions and obtained the Bayes estimates under these loss functions. Here is one example of the asymmetric loss function defined for the positive values of the parameter. If $\hat{\beta}$ is an estimate of β then the DeGroot loss function is defined as:

$$L_{DeGroot}(\hat{\beta},\beta) = \left(\frac{\beta-\hat{\beta}}{\hat{\beta}}\right),$$

The Bayes estimator and posterior risks under DLF can be derived by using following formulae:

$$\hat{\beta}_{DeGroot} = \frac{E_{(\beta|x)}(\beta^2)}{E_{(\beta|x)}(\beta)} \;,\; \text{Risk}\left(\hat{\beta}_{DeGroot}\right) = \frac{Var_{(\beta|x)}(\beta)}{E_{(\beta|x)}(\beta^2)}.$$

3.2. Linex Loss Function (LLF)

The linear-exponential loss function (LINEX) has been introduced by Varian [14], and various authors as (see Basu and Ebrahimi [15] and Soliman [16-17]) have used this loss function in different estimation problems. Under the assumption that the minimal loss occurs at $\hat{\beta} = \beta$, the Linex loss function for $L_{Linex}(\hat{\beta}, \beta)$ can be expressed as

$$L_{Linex}\left(\hat{\beta}, \beta\right) = b\left\{expc^*\left(\hat{\beta} - \beta\right) - c^*\left(\hat{\beta} - \beta\right) - 1\right\}, \quad c^* \neq 0\,, \;\; b > 0.$$

Without loss of generality, we assume that $b = 1$. Under the Linex loss function the Bayes estimator and posterior risk are defines as:

$$\hat{\beta}_{Linex} = \frac{-1}{c^*}\ln\{E(\exp(-c^*\beta))\}, \qquad \text{Risk}\left(\hat{\beta}_{Linex}\right) = \ln E(c^*\beta) + c^* E(\beta).$$

3.3. General Entropy Loss Function (GELF)

The linex loss function is suitable for the estimation of the location parameter but not for the estimation of the scale parameter and other parametric functions. Calabria and Pulcini [18] suggested the general entropy loss function (GELF) for estimation these quantities which can be defined it as:

$$L_{GELF}(\hat{\beta}, \beta) = b\left\{\left(\frac{\hat{\beta}}{\beta}\right)^p - pln\left(\frac{\hat{\beta}}{\beta}\right) - 1\right\}, b > 0\,, p \neq 0,$$

which has a minimum at $\hat{\beta} = \beta$. Without loss of generality, we assume that $b = 1$. This loss is a generalization of the entropy loss function that has been used by several authors taking the shape parameter $p = 1$. This general version allows different shapes of loss function when $p > 0$ and for $\hat{\beta} > 0$, i.e. a positive error causes more serious consequences than a negative error. The Bayes estimator of β under the general entropy loss is

$$\hat{\beta}_{GELF} = \left[E_{(\beta|x)}(\beta^{-p})\right]^{-\frac{1}{p}}.$$

Provided that $E_{(\beta|x)}(\beta^{-p})$ exists and is finite. The posterior risk under GELF can be defined as:

$$Risk(\hat{\beta}_{GELF}) = \ln\{E_{(\beta|x)}(\beta^{-p})\} + pE\{ln\beta\}.$$

4. Bayesian Analysis

This section includes the derivations of the expressions for the shape parameter of the Kumaraswamy distribution under different informative priors using DLF, LLF and GELF.

4.1. The Posterior Distribution and Estimators under Inverse Levy Prior

It is assumed that the prior distribution of β is Inverse Levy distribution with hyperparameter 'l' which is given below;

$$p_1(\beta) = \sqrt{\frac{l}{2\pi}}\ \beta^{-\frac{1}{2}} \exp\left(\frac{-l\beta}{2}\right),\ \beta\ , l > 0. \tag{5}$$

Now the posterior distribution of β given data is:

$$p_1(\beta\,|\mathbf{x}) \propto L(\mathbf{x};\,\beta)p_1(\beta), \tag{6}$$

$$p_1(\beta\,|\mathbf{x}) \propto \beta^{r+\frac{1}{2}} \exp\left\{-\beta\left(\frac{l}{2} - \left(\sum_{i=1}^{r} \ln\left(1 - x_{(i)}^{\alpha}\right) - (n-r)\ln\left(1 - x_{(r)}^{\alpha}\right)\right)\right\}, \tag{7}$$

This is the density kernel of the Gamma distribution with parameters $\upsilon_{1k} = r + \frac{1}{2}$, and $\gamma_{1k} = \frac{l}{2} + \varphi$. Where φ is defined above.

4.1.1. Estimation of $\boldsymbol{\beta}$: The Bayes estimator $\hat{\beta}_{DeGroot}$ of β relative to DeGroot loss function is given by:

$$\hat{\beta}_{DeGroot} = \frac{r + \frac{3}{2}}{\left\{\frac{l}{2} - \left(\sum_{i=1}^{r} \ln\left(1 - x_{(i)}^{\alpha}\right) - (n-r)\ln\left(1 - x_{(r)}^{\alpha}\right)\right\}} = \frac{r + \frac{3}{2}}{\left\{\frac{l}{2} + \varphi\right\}},$$

And posterior risk is: $\rho\left(\hat{\beta}_{DeGroot}\right) = \frac{1}{r+\frac{3}{2}}$.

The Bayes estimator $\hat{\beta}_{Linex}$ of β relative to Linex loss function is given by:

$$\hat{\beta}_{Linex} = \frac{-1}{c^*} ln\left\{\frac{\frac{l}{2} + \varphi}{c^* + \frac{l}{2} + \varphi}\right\}^{\left(r+\frac{1}{2}\right)},$$

And posterior risk under Linex loss function is given by:

$$\rho\left(\hat{\beta}_{Linex}\right) = ln\left\{\frac{\frac{l}{2} + \varphi}{c^* + \frac{l}{2} + \varphi}\right\}^{\left(r+\frac{1}{2}\right)} + c^*\left\{\frac{\frac{1}{2} + \varphi}{\frac{l}{2} + \varphi}\right\}.$$

The Bayes estimator $\hat{\beta}_{GELF}$ of β relative to GELF is given by:

$$\hat{\beta}_{GELF} = \left\{\frac{\left(\frac{l}{2} + \varphi\right)^{p} \Gamma\left(r + \frac{1}{2} - p\right)}{\Gamma\left(r + \frac{1}{2}\right)}\right\}^{-\frac{1}{p}},$$

And posterior risk under GELF given by:

$$\rho\left(\hat{\beta}_{GELF}\right) = ln\left\{\frac{\left(\frac{l}{2} + \varphi\right)^{p} \Gamma\left(r + \frac{1}{2} - p\right)}{\Gamma\left(r + \frac{1}{2}\right)}\right\} + p\left\{\psi\left(r + \frac{1}{2}\right) - ln\left(\frac{l}{2} + \varphi\right)\right\}.$$

Where $\psi(x) = \frac{\acute{\Gamma}(x)}{\Gamma(x)}$ is defined as digamma function.

4.2. The Posterior Distribution and Estimators under Gamma Prior

Under the assumption that the prior distribution of β is Gamma distribution with shape and scale parameter 'a' and 'b' respectively and it has the pdf

$$p_{2(\beta)} = \frac{b^a}{\Gamma(a)} \beta^{a-1}\ exp(-b\beta),\ \ a, b > 0, \tag{8}$$

The posterior distribution of β given data is:

$$p_2\ (\beta\ |\mathbf{x}) \propto \beta^{r+a+\frac{1}{2}} \exp\{-\beta\big(b - \big(\textstyle\sum_{i=1}^{r} \ln\,(1 - x_{(i)}^{\alpha}\big) - (n-r)\ln\big(1 - x_{(r)}^{\alpha}\big)\big)\}, \tag{9}$$

which is the density kernel of Gamma distribution with parameters $\upsilon_{2k} = r + a$, and $\gamma_{2k} = b +\ \varphi$.

4.2.1. Estimation of $\boldsymbol{\beta}$: The Bayes estimator $\hat{\beta}_{DeGroot}$ of β relative to DeGroot loss function is given by:

$$\hat{\beta}_{DeGroot} = \frac{r + a + 1}{\{b - \big(\sum_{i=1}^{r} \ln\,(1 - x_{(i)}^{\alpha}\big) - (n-r)\ln\big(1 - x_{(r)}^{\alpha}\big)\}} = \frac{r + a + 1}{\{b + \varphi\}}\ ,$$

And posterior risk is: $\rho\big(\hat{\beta}_{DeGroot}\big) = \frac{1}{r+a+1}$.

The Bayes estimator $\hat{\beta}_{Linex}$ of β relative to Linex loss function is given by:

$$\hat{\beta}_{Linex} = \frac{-1}{c^*} ln\left\{\frac{b + \varphi}{c^* + b + \varphi}\right\}^{(r+a)},$$

And posterior risk under Linex loss function is given by:

$$\rho\big(\hat{\beta}_{Linex}\big) = ln\left\{\frac{b + \varphi}{c^* + b + \varphi}\right\}^{(r+a)} + c^*\left\{\frac{r + \varphi}{b + \varphi}\right\}.$$

The Bayes estimator $\hat{\beta}_{GELF}$ of β relative to GELF loss function is given by:

$$\hat{\beta}_{GELF} = \left\{\frac{(b + \varphi)^p \Gamma(r + a - p)}{\Gamma(r + a)}\right\}^{-\frac{1}{p}},$$

And posterior risk under GELF given by:

$$\rho\big(\hat{\beta}_{GELF}\big) = ln\left\{\frac{(b + \varphi)\Gamma(r + a - p)}{\Gamma(r + a)}\right\} + p\{\psi(r + a) - ln(b +\ \varphi)\}.$$

Where $\psi(x) = \frac{\acute{\Gamma}(x)}{\Gamma(x)}$ is defined as digamma function.

4.3. The Posterior Distribution and Estimators under Mixture of Gamma and Jeffreys Prior

The mixture of Gamma and Jeffreys prior is defined as:

$$p_3\,(\beta\,|\mathbf{x}) = w\frac{b_1^{a_1}}{\Gamma(a_1)}\beta^{a_1-1}\;exp(-b_1\beta) + (1-w)\frac{1}{\beta}\;,\quad a_1, b_1 > 0\,,\;0 < w < 1. \tag{10}$$

The posterior distribution of β given data is:

$$p_3\,(\beta\,|\mathbf{x}) = \frac{w\frac{b_1^{a_1}}{\Gamma(a_1)}\beta^{r+a_1-1}\;exp\{-(b_1+\varphi)\}+(1-w)\beta^{r-1}exp\{-(\varphi)\}}{w\frac{b_1^{a_1}\Gamma(r+a_1)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1}}+(1-w)\frac{\Gamma(r)}{\varphi^r}}. \tag{11}$$

4.3.1. Estimation of β: The Bayes estimator $\hat{\beta}_{DeGroot}$ of β relative to DeGroot loss function is given by:

$$\hat{\beta}_{DeGroot} = \frac{w\frac{b_1^{a_1}\Gamma(r+a_1\;+2)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1+2}} + (1-w)\frac{\Gamma(r+2)}{\varphi^{r+2}}}{w\frac{b_1^{a_1}\Gamma(r+a_1\;+1)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1+1}} + (1-w)\frac{\Gamma(r+1)}{\varphi^{r+1}}}\,,$$

And posterior risk

$$\rho\left(\hat{\beta}_{DeGroot}\right) = 1 - \frac{\left\{w\frac{b_1^{a_1}\Gamma(r+a_1\;+1)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1+1}}+(1-w)\frac{\Gamma(r+1)}{\varphi^{r+1}}\right\}^2}{\left\{w\frac{b_1^{a_1}\Gamma(r+a_1\;)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1}}+1-w)\frac{\Gamma(r)}{\varphi^r}\right\}\left\{w\frac{b_1^{a_1}\Gamma(r+a_1\;+2)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1+2}}+(1-w)\frac{\Gamma(r+2)}{\varphi^{r+2}}\right\}}.$$

The Bayes estimator $\hat{\beta}_{Linex}$ of β relative to Linex loss function is given by:

$$\hat{\beta}_{Linex} = \frac{-1}{c^*}ln\left\{\frac{w\frac{b_1^{a_1}\Gamma(r+a_1\;)}{\Gamma(a_1)\{b_1+c^*+\varphi\}^{r+a_1}} + (1-w)\frac{\Gamma(r)}{(c^*+\varphi)^r}}{\left\{w\frac{b_1^{a_1}\Gamma(r+a_1\;)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1}} + (1-w)\frac{\Gamma(r)}{(\varphi)^r}\right\}}\right\},$$

And posterior risk under Linex loss function is given by:

$$\rho\left(\hat{\beta}_{Linex}\right) = ln\left\{\frac{w\frac{b_1^{a_1}\Gamma(r+a_1\;)}{\Gamma(a_1)\{b_1+c^*+\varphi\}^{r+a_1}} + (1-w)\frac{\Gamma(r)}{(c^*+\varphi)^r}}{w\frac{b_1^{a_1}\Gamma(r+a_1\;)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1}} + (1-w)\frac{\Gamma(r)}{\varphi^r}}\right\} + c^*\left\{\frac{w\frac{b_1^{a_1}\Gamma(r+a_1\;+1)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1+1}} + (1-w)\frac{\Gamma(r+1)}{(\varphi)^{r+1}}}{w\frac{b_1^{a_1}\Gamma(r+a_1\;)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1}} + (1-w)\frac{\Gamma(r)}{\varphi^r}}\right\}.$$

The Bayes estimator $\hat{\beta}_{GELF}$ of β relative to GELF loss function is given by:

$$\hat{\beta}_{GELF} = \left\{ \frac{w \frac{b_1^{a_1}\Gamma(r+a_1 \ -p)}{\Gamma(a_1)\{\varphi\}^{r+a_1-p}} + (1-w)\frac{\Gamma(\mathrm{r}-\mathrm{p})}{(\varphi)^{r-p}}}{w \frac{b_1^{a_1}\Gamma(r+a_1 \ )}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1}} + (1-w)\frac{\Gamma(\mathrm{r})}{\varphi^r}} \right\}^{-(\frac{1}{p})},$$

And posterior risk under GELF given by:

$$\rho(\hat{\beta}_{GELF}) = \ln \left\{ \frac{w \frac{b_1^{a_1}\Gamma(r+a_1 \ -p)}{\Gamma(a_1)\{\varphi\}^{r+a_1-p}} + (1-w)\frac{\Gamma(\mathrm{r}-\mathrm{p})}{(\varphi)^{r-p}}}{w \frac{b_1^{a_1}\Gamma(r+a_1 \ )}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1}} + (1-w)\frac{\Gamma(\mathrm{r})}{\varphi^r}} \right\} + p\left[\frac{1}{w \frac{b_1^{a_1}\Gamma(r+a_1 \ )}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1}} + (1-w)\frac{\Gamma(\mathrm{r})}{\varphi^r}} \left\{ w \frac{b_1^{a_1}\Gamma(r+a_1 \ )}{\Gamma(a_1)\{\varphi\}^{r+a_1}} \left(\psi(r+a_1) - ln(b_1+\varphi)\right) \right\} + (1-w)\frac{\Gamma(\mathrm{r})}{\varphi^r}\{(\psi(r) - ln(\varphi))\} \right].$$

Where $\psi(\mathrm{x}) = \frac{\Gamma'(\mathrm{x})}{\Gamma(\mathrm{x})}$ is defined as digamma function.

5. Elicitation of Hyper-parameter

Elicitation is the process of talking out the expert knowledge about some unknown quantity of interest, or the probability of some future event, which can be used to supplement any numerical data we may have. If the expert in question does not have a statistical background, as often happens, translating their beliefs into a statistical form suitable for the use in our analyses can be a challenging task as described Dey [19]. Anyhow, prior elicitation is an important component of Bayesian statistics and yet to be invented. In any statistical analysis there will typically be some form of background knowledge available in addition to data at hand. There are various methods of elicitation in literature. Here we have used the method based on the prior predictive distribution, which is developed by Aslam [20].

5.1. Method of Elicitation Through Prior Predictive Probabilities

The prior predictive removes the uncertainty in the parameters to reveal a distribution for the data point only. We suppose that prior predictive probabilities satisfy the laws of probability because this law ensure the expert would be consistent in eliciting the probabilities and some inconsistencies may arise which are not very serious. A function $\xi(a_1, a_2)$ is defined in such a way that the hyper-parameters a_1, and a_2 are to be chosen by minimizing this function

$$\xi(a_1, a_2) = \min_{a_1, a_2} \sum\nolimits_{n_{12}}^{r_{12}} \left\{ \frac{p(n_{12}) - p_0(n_{12})}{p(n_{12})} \right\}^2,$$

where $p(n_{12})$ denote the prior predictive probabilities characterized by the hyper-parameters

a_1, and a_2 and $p(n_{12})$ denote the elicited prior predictive probabilities. If the prior predictive distribution is symmetrical the hyper-parameters a_1, and a_2 are equal $(say\ c)$, so above equation becomes where $p(n_{12})$ and $p_0(n_{12})$ are the symmetrical prior predictive probabilities characterized by the hyper-parameters c and the elicited prior predictive probabilities respectively. By solving the above equations simultaneously by applying 'PROC SYSLIN' of the SAS package for eliciting the required hyper-parameters.

5.2. Elicitation through Prior Predictive Probabilities Assuming Inverse Levy Prior

The prior predictive distribution is defined as:

$$p(y) = \int_0^\infty p(y|\beta)p(\beta)\,d\beta \tag{12}$$

Using (5) and (7) the prior predictive distribution can be written as:

$$p(y) = \int_0^\infty \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\sqrt{\frac{l}{2\pi}}\,\beta^{-\frac{1}{2}}\exp\left(\frac{-l\beta}{2}\right)d\beta\,,$$

After simplification

$$p(y) = \sqrt{\frac{l}{2}}\left\{\frac{\alpha y^{\alpha-1}}{2(1-y^{\alpha})\left\{\frac{l}{2}-\ln(1-y^{\alpha})\right\}^{3/2}}\right\}. \tag{13}$$

By the mentioned method of elicitation we obtain the following value of hyper-parameter $l = 0.079592$.

5.3. Elicitation through Prior Predictive Probabilities Assuming Gamma Prior

The equation of prior predictive using Gamma prior is:

$$p(y) = \frac{b^{a}}{\Gamma(a)}\int_0^\infty \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\beta^{a-1}\ exp(-b\beta)d\beta \tag{14}$$

Which simplifies to:

$$p(y) = \frac{ab^{a}\alpha y^{\alpha-1}}{(1-y^{\alpha})\{b-\ln(1-y^{\alpha})\}^{(a+1)}} \tag{15}$$

Hence the obtained values of hyper-parameters $a = 0.245642$, $b = 0.04642$.

5.4. Elicitation Assuming Mixture of Gamma and Jeffreys Priors

The equation of prior predictive using Gamma prior is:

$$p(y) = w\frac{b_1^{a_1}}{\Gamma(a_1)}\int_0^\infty \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\ \beta^{a_1-1}\ exp(-b_1\beta)d\beta + (1-w)\int_0^\infty \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\ \frac{1}{\beta}\,d\beta$$

$$p(y) = w\frac{b_1^{a_1}}{\Gamma(a_1)}\left\{\frac{\alpha y^{\alpha-1}}{(1-y^{\alpha})}\right\}\frac{\Gamma(a_1+1)}{\{b_1-\ln(1-y^{\alpha})\}^{(a_1+1)}} + (1-w)\left\{\frac{\alpha y^{\alpha-1}}{(1-y^{\alpha})}\right\}\left\{\frac{-1}{ln(1-y^{\alpha})}\right\} \tag{16}$$

According to mentioned method the elicited values of hyper-parameters $a_1 = 0.199343, b_1 = 1.227808$.

6. Posterior Predictive Distributions

The predictive distribution contains the information about the independent future random observation given preceding observations. Bansal [21] have given a great detailed discussion about the posterior predictive distribution.

6.1. Posterior Distribution and Posterior Predictive Intervals Assuming Inverse Levy Prior

The posterior predictive distribution of the future observation $y = x_{n+1}$ is:

$$p_1\,(y\,|\mathbf{x}) = \int_0^\infty p(y|\beta)p(\beta|\mathbf{x})d\beta \tag{17}$$

$$p_1\,(y|\mathbf{x}) = \left\{\frac{\alpha(r+\frac{1}{2})(\frac{l}{2}+\varphi)^{(r+\frac{1}{2})}y^{\alpha-1}}{(1-y^\alpha)\left\{\frac{l}{2}-\left(\sum_{i=1}^r \ln\left(1-x_{(i)}^\alpha\right)-(n-r)\ln\left(1-x_{(r)}^\alpha\right)-\ln(1-y^\alpha)\right\}^{r+3/2}}\right\},$$

$$p_1\,(y\,|\mathbf{x}) = \left\{\frac{\alpha(r+\frac{1}{2})(\frac{l}{2}+\varphi)^{(r+\frac{1}{2})}y^{\alpha-1}}{(1-y^\alpha)\left\{\frac{l}{2}+\varphi-\ln(1-y^\alpha)\right\}^{r+3/2}}\right\} \tag{18}$$

And posterior predictive intervals are

$$\int_0^L p(y|\mathrm{x})dy = \frac{k}{2} = \int_u^1 p(y|\mathrm{x})dy \tag{19}$$

$$1-\left\{\frac{\frac{l}{2}+\varphi}{\frac{l}{2}+\varphi-\ln(1-L^\alpha)}\right\}^{r+\frac{1}{2}} = \frac{k}{2},$$

$$\left\{\frac{\frac{l}{2}+\varphi}{\frac{l}{2}+\varphi-\ln(1-U^\alpha)}\right\}^{r+\frac{1}{2}} = \frac{k}{2}.$$

6.2. Posterior Distribution and Posterior Predictive Intervals Assuming Inverse Levy Prior

The posterior predictive distribution of the future observation $y = x_{n+1}$ is:

$$p_2\,(y\,|\mathbf{x}) = \left\{\frac{\alpha(r+a)(b+\varphi)^{(r+a)}y^{\alpha-1}}{(1-y^\alpha)\{b+\varphi-\ln(1-y^\alpha)\}^{(r+a+1)}}\right\} \tag{20}$$

And predictive intervals are:

$$1-\left\{\frac{b+\varphi}{\frac{l}{2}+\varphi-\ln(1-L^\alpha)}\right\}^{r+a} = \frac{k}{2},\ \left\{\frac{b+\varphi}{b+\varphi-\ln(1-U^\alpha)}\right\}^{r+a} = \frac{k}{2}.$$

6.3. Posterior Distribution and Posterior Predictive Intervals Assuming Mixture of Gamma and Jeffreys Prior

The posterior predictive distribution of the future observation $y = x_{n+1}$ is:

$$p_3\,(y\,|\mathbf{x}) = \frac{1}{\left(w\frac{b_1^{a_1}\Gamma(r+a_1\,)}{\Gamma(a_1)(b_1+\varphi)^{r+a_1}}+(1-w)\frac{\Gamma(r)}{\varphi^r}\right)}\left\{w\frac{\alpha y^{\alpha-1}b_1^{a_1}\Gamma(r+a_1+1)}{\Gamma(a_1)(1-y^{\alpha})\{b_1+\varphi-\ln(1-y^{\alpha})\}^{r+a_1+1}}+(1-w)\frac{(\alpha y^{\alpha-1})\Gamma(r)}{(1-y^{\alpha})\varphi^r}\right\}$$

And predictive intervals are:

$$\left\{\frac{1}{w\frac{b_1^{a_1}\Gamma(r+a_1\,)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1}}+(1-w)\frac{\Gamma(r)}{\varphi^r}}\right\}\left[\left\{\frac{\frac{wb_1^{a_1}\Gamma(r+a_1\,)}{\Gamma(a_1)}}{(b_1+\varphi)^{r+a_1}}-\frac{\frac{wb_1^{a_1}\Gamma(r+a_1\,)}{\Gamma(a_1)}}{(b_1+\varphi-\ln(1-L^{\alpha}))^{r+a_1}}\right\}\right.$$
$$\left.+\left\{\frac{(1-w)\Gamma(r)}{(\varphi)^r}-\frac{(1-w)\Gamma(r)}{(\varphi-\ln(1-L^{\alpha}))^r}\right\}\right]=\frac{k}{2}\,,$$

$$\left\{\frac{1}{w\frac{b_1^{a_1}\Gamma(r+a_1\,)}{\Gamma(a_1)\{b_1+\varphi\}^{r+a_1}}+(1-w)\frac{\Gamma(r)}{\varphi^r}}\right\}\left[\left\{\frac{\frac{wb_1^{a_1}\Gamma(r+a_1\,)}{\Gamma(a_1)}}{(b_1+\varphi-\ln(1-U^{\alpha}))^{r+a_1}}\right\}+\left\{\frac{(1-w)\Gamma(r)}{(\varphi-\ln(1-U^{\alpha}))^r}\right\}\right]=\frac{k}{2}\,.$$

7. Simulation Study

In order to assess the statistical performances of these estimates, we conducted a simulation study. The risks using generated random samples of different sizes are computed for each estimator. The behavior of the different estimators under different censoring schemes and prior distribution has been examined. The term different censoring scheme means different values of n and r. We considered six censoring scheme. For all the censoring schemes, we have used $\alpha = 2$, $w = 0.4, 0.6$ and $\beta = 0.5, 1, 1.5$. We have taken three informative priors and the elicited values of the hyper-parameters have been utilized. Inverse transformation method has been used for data generation. As one data set does not help to clarify performance of the estimator, so we have computed the average Bayes estimates along the corresponding posterior risks based on 10,000 data generation replications. In order to draw comparisons, we have computed MLEs, and UMVUE estimates. The results are reported in Table 1-23.

Table 1. B.Es and P.Rs (given in parentheses) under DeGroot Loss Function with True Value of $\alpha = 2$.

Sample Scheme (n, r)	Inverse Levy Prior $\beta = 0.5$	Inverse Levy Prior $\beta = 1$	Inverse Levy Prior $\beta = 1.5$	Gamma Prior $\beta = 0.5$	Gamma Prior $\beta = 1$	Gamma Prior $\beta = 1.5$
(20, 15)	0.5223672 (0.060606)	0.957187 (0.060606)	1.343090 (0.060606)	0.513211 (0.061555)	0.943254 (0.061555)	1.324640 (0.061555)
(20, 18)	0.548105 (0.051282)	1.059710 (0.051282)	1.539560 (0.051282)	0.540782 (0.0519598)	1.04724 (0.0519598)	1.517110 (0.0519598)
(50, 38)	0.479362 (0.051282)	0.885796 (0.051282)	1.247850 (0.051282)	0.476299 (0.0254805)	0.880147 (0.0254805)	1.240690 (0.0254805)
(50, 45)	0.507615 (0.021505)	0.982013 (0.021505)	1.425570 (0.021505)	0.504671 (0.0216237)	0.976548 (0.0216237)	1.419660 (0.0216237)
(100,75)	0.463931 (0.013072)	0.853233 (0.013072)	1.202120 (0.013072)	0.461829 (0.0131155)	0.849809 (0.0131155)	1.195330 (0.0131155)
(100, 90)	0.494693 (0.010929)	0.955991 (0.010929)	1.3948070 (0.010929)	0.492423 (0.0109594)	0.953382 (0.0109594)	1.389250 (0.0109594)

Table 2. B.Es and P.Rs (given in parentheses) using DLF under mixture prior with true value of $\alpha = 2$.

Sample Scheme (n, r)	$w = 0.4$, $\beta = 0.5$	$w = 0.4$, $\beta = 1$	$w = 0.4$, $\beta = 1.5$	$w = 0.6$, $\beta = 0.5$	$w = 0.6$, $\beta = 1$	$w = 0.6$, $\beta = 1.5$
(20, 15)	0.507468 (0.0624889)	0.92872 (0.0625917)	1.31060 (0.0626691)	0.506544 (0.0624714)	0.924475 (0.0626817)	1.300590 (0.0628481)
(20, 18)	0.536984 (0.0526291)	1.03314 (0.0527156)	1.503140 (0.0527737)	0.534891 (0.0526214)	1.03289 (0.0528023)	1.502310 (0.0529304)
(50, 38)	0.474754 (0.0256381)	0.874292 (0.0256562)	1.23099 (0.0256707)	0.472473 (0.025634)	0.871732 (0.0256707)	1.23011 (0.0257021)
(50, 45)	0.502789 (0.0217379)	0.970121 (0.0217533)	1.413440 (0.0217646)	0.502342 (0.0217358)	0.968260 (0.0217673)	1.41112 (0.0217924)
(100,75)	0.460256 (0.013157)	0.847133 (0.0131617)	1.192490 (0.0131657)	0.45559909 (0.0131558)	0.846621 (0.0131654)	1.19183 (0.0131738)
(100, 90)	0.0492171 (0.0109886)	0.950981 (0.0109926)	1.385610 (0.0109956)	0.491184 (0.010988)	0.949595 (0.0109961)	1.38243 (0.0110028)

Table 3. B.Es and P.Rs (given in parentheses) using LLF under inverse levy prior for $\alpha = 2$.

Sample Scheme (n, r)	$\beta = 0.5$, $c = -1$	$\beta = 0.5$, $c = 1$	$\beta = 1$, $c = -1$	$\beta = 1$, $c = 1$	$\beta = 1.5$, $c = -1$	$\beta = 1.5$, $c = 1$
(20, 15)	0.498663 (0.00841743)	0.484514 (0.00810982)	0.930409 (0.0288832)	0.877454 (0.026666)	1.32628 (0.0580123)	1.21425 (0.0511996)
(20, 18)	0.529438 (0.00794332)	0.512834 (0.00757832)	1.03694 (0.0300978)	0.977033 (0.0276323)	1.52877 (0.0647816)	1.40734 (0.0577436)
(50, 38)	0.470033 (0.00292346)	0.464868 (0.00288053)	0.872496 (0.0100124)	0.853371 (0.00971944)	1.23536 (0.0199893)	1.19979 (0.0192447)
(50, 45)	0.498813 (0.00278399)	0.495048 (0.00276217)	0.971018 (0.010505)	0.94845 (0.0101543)	1.42089 (0.0223807)	1.37554 (0.0214054)
(100,75)	0.459178 (0.0014091)	0.457049 (0.00140191)	0.847789 (0.00478847)	0.837358 (0.00470725)	1.193460 (0.00947207)	1.17717 (0.00931016)
(100, 90)	0.490892 (0.00134288)	0.488135 (0.00133266)	0.950184 (0.00501987)	0.941076 (0.00495763)	1.38725 (0.0106771)	1.36626 (0.0104624)

Table 4. B.Es and P.Rs (given in parentheses) under Linex loss function with true value of $\alpha = 2$.

Sample Schem (n, r)	Gamma Prior $\beta = 0.5$ $c = -1$	$c = 1$	$\beta = 1$ $c = -1$	$c = 1$	$\beta = 1.5$ $c = -1$	$c = 1$
(20, 15)	0.492603 (0.0082454	0.473647 (0.0079768	0.916385 (0.0280793	0.854044 (0.025965)	1.31335 (0.0570688	1.18290 (0.0500441
(20, 18)	0.525584 (0.0078172	0.503617 (0.0074985	1.02671 (0.0295571	0.959125 (0.0273394	1.51789 (0.0638424	1.3777609 (0.056521)
(50, 38)	0.471798 (0.0029251	0.458301 (0.0028565	0.872721 (0.0099480	0.843626 (0.0096884	1.23637 (0.0198911	1.18259 (0.0190749
(50, 45)	0.499284 (0.0027670	0.488418 (0.0027383	0.974323 (0.0104907	0.936249 (0.0100771	1.42132 (0.0222253	1.36204 (0.0213863
(100,75)	0.459761 (0.0013987	0.45108 (0.0013880	0.849609 (0.0047619	0.828658 (0.0046863	1.19636 (0.0094238	1.16505 (0.0092736
(100, 90)	0.492625 (0.0013386	0.0482953 (0.0013252	0.953022 (0.0049975	0.932853 (0.0049503	1.39315 (0.0106599	1.35590 (0.0104716

Table 5. B.Es and P.Rs (given in parentheses) under Linex loss function for $\alpha = 2,\ w = 0.4$.

Sample Scheme (n, r)	The Mixture of Gamma and Jeffreys Prior $\beta = 0.5$ $c = -1$	$c = 1$	$\beta = 1$ $c = -1$	$c = 1$	$\beta = 1.5$ $c = -1$	$c = 1$
(20, 15)	0.482346 (0.00813824)	0.467966 (0.00782097)	0.896889 (0.0277999)	0.848112 (0.025801)	1.28372 (0.0563583)	1.17239 (0.0494711)
(20, 18)	0.515587 (0.00773397)	0.499387 (0.00738889)	1.00487 (0.0290983)	0.951278 (0.0269864)	1.49163 (0.0636077)	1.37004 (0.0563985)
(50, 38)	0.463634 (0.00288134)	0.459137 (0.00284595)	0.861566 (0.00990067)	0.842775 (0.00961196)	1.21870 (0.0197317)	1.18184 (0.0189431)
(50, 45)	0.49456 (0.00276637)	0.487976 (0.00271148)	0.958478 (0.0103476)	0.940137 (0.0100953)	1.40700 (0.0222186)	1.36034 (0.021193)
(100,75)	0.455279 (0.0013943)	0.453169 (0.00138714)	0.84088 (0.00474551)	0.831742 (0.00467676)	1.18602 (0.00942176)	1.16653 (0.00920836)
(100, 90)	0.488371 (0.00133634)	0.485623 (0.00132626)	0.946253 (0.00500902)	0.934177 (0.00491459)	1.380120 (0.0106332)	1.35911 (0.0104185)

Table 6. B.Es and P.Rs (given in parentheses) under Linex loss function for $\alpha = 2,\ w = 0.6$.

Sample Schem (n, r)	The Mixture of Gamma and Jeffreys Prior $\beta = 0.5$ $c = -1$	$c = 1$	$\beta = 1$ $c = -1$	$c = 1$	$\beta = 1.5$ $c = -1$	$c = 1$
(20, 15)	0.481602 (0.0081380	0.464832 (0.0077051	0.895483 (0.027771)	0.842401 (0.0254717	1.28072 (0.0563467	1.17230 (0.0496521
(20, 18)	0.512208 (0.0076418	0.498795 (0.0073653	1.00096 (0.0289456	0.949719 (0.0269703	1.48234 (0.0629169	1.36191 (0.0558605
(50, 38)	0.464148 (0.0028864	0.457419 (0.0028261	0.858787 (0.0098380	0.840078 (0.0095535	1.21701 (0.0197146	1.17940 (0.018898)
(50, 45)	0.494753 (0.0027658	0.486904 (0.0026986	0.957674 (0.0103338	0.936725 (0.100249)	1.40269 (0.0221288	1.35889 (0.0211803
(100,75)	0.45554891 (0.0013918	0.452497 (0.0013827	0.840179 (0.0047385	0.828747 (0.0046439	1.18524 (0.0094140	1.16577 (0.0092026
(100, 90)	0.487578 (0.0013323	0.484344 (0.0013193	0.945219 0.00499944	0.932723 0.00490098	1.37902 (0.0106232	1.3555814 (0.0104112

Table 7. B.Es and P.Rs (given in parentheses) under GELF with true value of $\alpha = 2$, $\beta = 0.5$.

Sample Scheme	Inverse Levy Prior			
(n, r)	$p = -1$	$p = -0.5$	$p = 0.5$	$p = 1$
(20, 15)	0.491559 (0.0326048)	0.483291 (0.00823927)	0.466864 (0.00842431)	0.458175 (0.0340866)
(20, 18)	0.517946 (0.0272704)	0.512747 (0.00687929)	0.498929 (0.00700782)	0.491445 (0.0282994)
(50, 38)	0.466667688 (0.0130432)	0.465932 (0.00327495)	0.458744 (0.0033038)	0.455843 (0.0132741)
(50, 45)	0.49711 (0.0110293)	0.494317 (0.00276743)	0.490536 (0.00278801)	0.48582 (0.0111939)
(100,75)	0.457665 (0.00663714)	0.455426 (0.00166295)	0.452349 (0.00167039)	0.451275 (0.0066964)
(100, 90)	0.488858 (0.00553304)	0.488184 (0.00138631)	0.485071 (0.00139145)	0.483813 (0.00557619)

Table 8. B.Es and P.Rs (given in parentheses) under GELF with true value of $\alpha = 2$, $\beta = 1$

Sample Scheme	Inverse Levy Prior			
(n, r)	$p = -1$	$p = -0.5$	$p = 0.5$	$p = 1$
(20, 15)	0.899114 (0.0326048)	0.887119 (0.00823927)	0.861040 (0.00842431)	0.843621 (0.0340866)
(20, 18)	1.00441 (0.0272704)	0.990952 (0.00687929)	0.965025 (0.00700782)	0.948209 (0.0282994)
(50, 38)	0.862296 (0.01330432)	0.859129 (0.00327495)	0.846386 (0.0033038)	0.84334 (0.0132741)
(50, 45)	0.960749 (0.0110293)	0.954825 (0.00276743)	0.941448 (0.00278801)	0.939777 (0.0111939)
(100,75)	0.841285 (0.00663714)	0.839273 (0.00166295)	0.83265 (0.00167036)	0.829537 (0.0066964)
(100, 90)	0.94491 (0.00553504)	0.941943 (0.00138631)	0.937604 (0.00139145)	0.934865 (0.00557619)

Table 9. B.Es and P.Rs (given in parentheses) under GELF with true value of $\alpha = 2$, $\beta = 1.5$.

Sample Scheme	Inverse Levy Prior			
(n, r)	$p = -1$	$p = -0.5$	$p = 0.5$	$p = 1$
(20, 15)	1.26494 (0.0326048)	1.24348 (0.00823927	1.203800 (0.00842431)	1.182100 (0.0340866)
(20, 18)	1.46243 (0.0272704)	1.44327 (0.00687929)	1.40176 (0.00700782)	1.38047 (0.0282994)
(50, 38)	1.21638 (0.0130432)	1.20923 (0.00327495)	1.19259 (0.0033038)	1.18650 (0.0132741)
(50, 45)	1.39394 (0.0110293)	1.38936 (0.00276743)	1.37038 (0.00278801)	1.36308 (0.0111939
(100,75)	1.1853 (0.00663714)	1.18055 (0.00166295)	1.17208 (0.00167036)	1.16826 (0.0066964)
(100, 90)	1.37723 (0.00553504)	1.37211 (0.00138631)	1.36417 (0.00139145)	1.35979 (0.00557619)

Table 10. B.Es and P.Rs (given in parentheses) under GELF with true value of $\alpha = 2$, $\beta = 0.5$.

Sample Scheme (n,r)	Gamma Prior $p=-1$	$p=-0.5$	$p=0.5$	$p=1$
(20, 15)	0.48285 (0.0331546)	0.474103 (0.00837972)	0.459771 (0.0085712)	0.450347 (0.034688)
(20, 18)	0.514633 (0.0276541)	0.50697 (0.00697693)	0.495184 (0.00710917)	0.486239 (0.0287127)
(50, 38)	0.463705 (0.0131304)	0.460887 (0.00329692)	0.45487 (0.00332616)	0.453232 (0.0133643)
(50, 45)	(0.494522 (0.0110915)	0.490275 (0.00278311)	0.486159 (0.00280391)	0.484361 (0.011258)
(100,75)	0.455828 (0.00665962)	0.4534 (0.0016686)	0.450127 (0.00167605)	0.449613 (0.00671928)
(100, 90)	0.48799932 (0.00555067)	0.48663 (0.00139023)	0.483489 (0.0013954)	0.482487 (0.00559205)

Table 11. B.Es and P.Rs (given in parentheses) under GELF with true value of $\alpha = 2$, $\beta = 1$.

Sample Scheme (n,r)	Gamma Prior $p=-1$	$p=-0.5$	$p=0.5$	$p=1$
(20, 15)	0.883975 (0.0331546)	0.870066 (0.00837972)	0.842106 (0.0085712)	0.827797 (0.034688)
(20, 18)	0.995888856 (0.027644541)	0.976029 (0.00697693)	0.94722 (0.00710917)	0.935958 (0.0287127)
(50, 38)	0.856988 (0.0131304)	0.852603 (0.00329692)	0.839909 (0.00332616)	0.835338 (0.0133643)
(50, 45)	0.953807 (0.0110915)	0.944951 (0.00278311)	0.939134 (0.00280391)	0.93523 (0.011258)
(100,75)	0.838775 (0.00665962)	0.836668 (0.0016686)	0.829493 (0.001677605)	0.826264 (0.00671928)
(100, 90)	0.942202 (0.00555067)	0.940126 (0.00139023)	0.934318 (0.0013954)	0.932843 (0.00559205)

Table 12. B.Es and P.Rs (given in parentheses) under GELF with true value of $\alpha = 2$, $\beta = 1.5$.

Sample Scheme (n,r)	Gamma Prior $p=-1$	$p=-0.5$	$p=0.5$	$p=1$
(20, 15)	1.24722 (0.0331546)	1.22213 (0.00837972)	1.17898 (0.0085712)	1.16003 (0.034688)
(20, 18)	1.44525 (0.0276541)	1.42747 (0.00697693)	1.37893 (0.00710917)	1.36172 (0.0287777127)
(50, 38)	1.20748 (0.0131304)	1.20094 (0.003229692)	1.18383 (0.00332616)	1.17514 (0.0133643)
(50, 45)	1.39192 (0.0110915)	1.38159 (0.00278311)	1.36623 (0.00280391)	1.35895 (0.011258)
(100,75)	1.18079 (0.00665962)	1.1779 (0.0016686)	1.16852 (0.00167605)	1.16440 (0.00671928)
(100, 90)	1.37415 (0.00555067)	1.36844 (0.00139023)	1.36038 (0.0013954)	1.35849 (0.00559205)

Table 13. B.Es and P.Rs (given in parentheses) under GELF for $\alpha = 2$, $\beta = 0.5$ and w=0.4.

Sample Scheme (n,r)	The Mixture of Gamma and Jeffreys Prior $p=-1$	$p=-0.5$	$p=0.5$	$p=1$
(20, 15)	0.472654 (0.0336939)	0.466044 (0.00851737)	0.450637 (0.0087149)	0.442359 (0.0352759)
(20, 18)	0.507347 (0.0280315)	0.499257 (0.00707289)	0.484947 (0.00720856)	0.478452 (0.0291179)
(50, 38)	0.462323 (0.0132139)	0.458028 (0.00331797)	0.452875 (0.00334757)	0.449466 (0.0134507)
(50, 45)	0.491411 (0.0111515)	0.489854 (0.00279821)	0.483468 (0.00281923)	0.480149 (0.0113197)
(100,75)	0.453559 (0.00668099)	0.452087 (0.00167396)	0.449863 (0.00168147)	0.447404 (0.00674102)
(100, 90)	0.486246 (0.00556563)	0.48576 (0.00139399)	0.483186 (0.00139918)	0.481442 (0.0056072)

Table 14. B.Es and P.Rs (given in parentheses) under GELF for $\alpha = 2$, $\beta = 0.5$ and w=0.6.

Sample Scheme (n,r)	The Mixture of Gamma and Jeffreys Prior $p=-1$	$p=-0.5$	$p=0.5$	$p=1$
(20, 15)	0.47216 (0.0336806)	0.466036 (0.0085139)	0.451243 (0.00871096)	0.440891 (0.0352584)
(20, 18)	0.506773 (0.0280255)	0.49773 (0.00707118)	0.485359 (0.00720669)	0.477356 (0.291093)
(50, 38)	0.0460514 (0.0132115)	0.457893 (0.00331738)	4.52156 (0.00334695)	0.448896 (0.0134481)
(50, 45)	(0.491292 (0.0111503)	0.488434 (0.0027979)	0.483098 (0.00281891)	0.480037 (0.0113183)
(100,75)	0.453074 (0.00668036)	0.451126 (0.0016738)	0.449779 (0.0016813)	0.447058 (0.00674036)
(100, 90)	0.485759 (0.0055653)	0.484622 (0.0013939)	0.481489 (0.001399)	0.480252 (0.00560688)

Table 15. B.Es and P.Rs (given in parentheses) under GELF for $\alpha = 2$, $\beta = 1$ and w=0.4.

Sample Scheme (n,r)	The Mixture of Gamma and Jeffreys Prior $p=-1$	$p=-0.5$	$p=0.5$	$p=1$
(20, 15)	0.870828 (0.0337513)	0.85796 (0.00853186)	0.829242 (0.00872967)	0.813559 (0.0353354)
(20, 18)	1.33174 (0.0281066)	1.30943 (0.00709192)	1.27646 (0.00722813)	1.25794 (0.0291972)
(50, 38)	0.85472 (0.0132234)	0.84371 (0.00332032)	0.833991 (0.00334995)	0.829687 (0.0134602)
(50, 45)	0.949678 (0.0111595)	0.945109 (0.00280022)	0.930824 (0.00282124)	0.929106 (0.0113278)
(100,75)	0.836131 (0.00668341)	0.83364 (0.00167457)	0.827351 (0.00168207)	0.826355 (0.00674346)
(100, 90)	0.941352 (0.00556766)	0.937686 (0.00139449)	0.931823 (0.00139969)	0.929061 (0.00560926)

Table 16. B.Es and P.Rs (given in parentheses) under GELF for $\alpha = 2$, $\beta = 1$ and w=0.6.

Sample Scheme (n,r)	The Mixture of Gamma and Jeffreys Prior $p=-1$	$p=-0.5$	$p=0.5$	$p=1$
(20, 15)	0.870169 (0.0337981)	0.854133 (0.00854326)	0.822869 (0.00874053)	0.812411 (0.0353796)
(20, 18)	0.975761 (0.0281236)	0.960809 (0.00709596)	0.933404 (0.00723175)	0.918216 (0.0292103)
(50, 38)	0.850047 (0.0132306)	0.843015 (0.00332214)	0.833156 (0.00335175)	0.827191 (0.0134673)
(50, 45)	0.949028 (0.0111667)	0.943209 (0.002802)	0.930383 (0.00282302)	0.927904 (0.0113348)
(100,75)	0.835547 (0.00668523)	0.831742 (0.00167502)	0.827654 (0.00168253)	0.824608 (0.00674526)
(100, 90)	0.938898 (0.00556944)	0.936803 (0.00139494)	0.931822 (0.00140014)	0.928893 (0.00561104)

Table 17. B.Es and P.Rs (given in parentheses) under GELF for $\alpha = 2$, $\beta = 1.5$ and w=0.4.

Sample Scheme (n,r)	The Mixture of Gamma and Jeffreys Prior $p=-1$	$p=-0.5$	$p=0.5$	$p=1$
(20, 15)	1.22482 (0.0337958)	1.20716 (0.00854321)	1.16011 (0.00874132)	1.14233 (0.0353833)
(20, 18)	1.42271 (0.0281127)	1.40531 (0.00709353)	1.36876 (0.00722982)	1.34512 (0.0292039)
(50, 38)	1.20206 (0.013231)	1.19433 (0.00332228)	1.17678 (0.00335192)	1.17070 (0.0134682)
(50, 45)	1.38115 (0.0111654)	1.37341 (0.00280171)	1.36115 (0.00282278)	1.35348 (0.011334)
(100,75)	1.17546 (0.00668543)	1.17033 (0.00167507)	1.16472 (0.00168258)	1.15967 (0.0067455)
(100, 90)	1.36924 (0.0055692)	1.36550 (0.00139488)	1.3570006 (0.00140008)	1.3587774 (0.00561085)

Table 18. B.Es and P.Rs (given in parentheses) under GELF for $\alpha = 2$, $\beta = 1.5$ and w=0.6.

Sample Scheme (n,r)	The Mixture of Gamma and Jeffreys Prior $p=-1$	$p=-0.5$	$p=0.5$	$p=1$
(20, 15)	1.22018 (0.0338929)	1.19853 (0.0085675)	1.16405 (0.00876632)	1.13777 (0.0354819)
(20, 18)	1.421317 (0.028198)	1.39447 (0.00711473)	1.36576 (0.00725166)	1.33477 (0.0292905)
(50, 38)	1.19903 (0.0132472)	1.19272 (0.00332635)	1.17429 (0.00335597)	1.16916 (0.0134845)
(50, 45)	1.37789 (0.0111797)	1.37203 (0.00280531)	1.35673 (0.00282639)	1.35152 (0.0113485)
(100,75)	1.17453 (0.00668958)	1.170235 (0.00167612)	1.16299 (0.00168362)	1.15901 (0.00674996)
(100, 90)	1.36918 (0.00557286)	1.36528 (0.0013958)	1.35683 (0.0014010)	1.3561 (0.00561453)

Table 19. MLEs and UMVUE Estimates and the corresponding MSE (given in parentheses) for $\alpha = 2$.

Sample (n, r)	MLE $\beta = 0.5$	MLE $\beta = 1$	MLE $\beta = 1.5$	UMVUE $\beta = 0.5$	UMVUE $\beta = 1$	UMVUE $\beta = 1.5$
(20, 15)	0.475345 (0.0233516)	0.87398 (0.0934066)	1.22228 (0.210165)	0.444239 (0.0192308)	0.815526 (0.0769231)	1.1458 (0.173077)
(20, 18)	0.509754 (0.0183824)	0.976033 (0.0735294)	1.42905 (0.165441)	0.482066 (0.015625)	0.928805 (0.0625)	1.35563 (0.140625)
(50, 38)	0.461138 (0.00750751)	0.852416 (0.03003)	1.20107 (0.0675676)	0.448657 (0.0069444)	0.830046 (0.027778)	1.16879 (0.0625)
(50, 45)	0.49235 (0.00621036)	0.950002 (0.0248414)	1.38374 (0.0558932)	0.480705 (0.00581395)	0.931067 (0.0232558)	1.355452 (0.0523327)
(100,75)	0.45465 (0.0035635)	0.836996 (0.014254)	1.17833 (0.0320715	0.44839 (0.00342466)	0.82676 (0.0136986)	1.16164 (0.0308219)
(100, 90)	0.486343 (0.00293667)	0.940435 (0.0117467)	1.36791 (0.026430)	0.480769 (0.00284091)	0.930414 (0.0113636)	1.35261 (0.0255682)

Table 20. 95% Posterior Predictive Intervals for β with true value of $\alpha = 2$.

Sample	Inverse Levy Prior					
(n,r)	$\beta = 0.5$		$\beta = 1$		$\beta = 1.5$	
	Lower	*Upper*	*Lower*	*Upper*	*Lower*	*Upper*
(20, 15)	0.230466	0.999936	0.170626	0.996103	0.143932	0.983851
(20, 18)	0.223751	0.999873	0.161489	0.992899	0.134446	0.973189
(50, 38)	0.232152	0.999895	0.171653	0.994816	0.1455005	0.980388
(50, 45)	0.225076	0.999812	0.162821	0.991482	0.133489	0.969132
(100, 75)	0.233232	0.999882	0.122991	0.954372	0.146158	0.9779897
(100, 90)	0.225838	0.999792	0.163413	0.991025	0.14845	0.981883

Table 21. 95% Posterior Predictive Intervals for β with true value of $\alpha = 2$.

Sample	Gamma Prior					
(n,r)	$\beta = 0.5$		$\beta = 1$		$\beta = 1.5$	
	Lower	*Upper*	*Lower*	*Upper*	*Lower*	*Upper*
(20, 15)	0.222495	0.999997	1.171206	0.997481	0.143264	0.98498
(20, 18)	0.215477	0.999890	0.160370	0.992520	0.133133	0.975113
(50, 38)	0.223061	0.999902	0.171248	0.994989	0.144399	0.980761
(50, 45)	0.224986	0.999985	0.161505	0.991783	0.132544	0.96956
(100, 75)	0.225870	0.999987	0.173173	0.994682	0.146129	0.981057
(100, 90)	0.2211030	0.997960	0.163619	0.991119	0.135844	0.98453

Table 22. 95% Posterior Predictive Intervals for β with true value of $\alpha = 2,\ w = 0.4$.

Sample	Mixture Prior					
(n,r)	$\beta = 0.5$		$\beta = 1$		$\beta = 1.5$	
	Lower	*Upper*	*Lower*	*Upper*	*Lower*	*Upper*
(20, 15)	0.230465	0.999956	0.170457	0.996767	0.156421	0.986935
(20, 18)	0.223407	0.999899	0.162726	0.994789	0.135804	0.976015
(50, 38)	0.232614	0.999906	0.171703	0.995119	0.982145	0.145812
(50, 45)	0.222496	0.999831	0.163756	0.999861	0.135427	0.971709
(100, 75)	0.233184	0.999890	0.173624	0.994810	0.146561	0.983270
(100, 90)	0.225298	0.999799	0.163536	0.991418	0.136066	0.978453

Table 23. 95% Posterior Predictive Intervals for β with true value of $\alpha = 2, w = 0.6.$

Sample	Mixture Prior					
(n, r)	$\beta = 0.5$		$\beta = 1$		$\beta = 1.5$	
	Lower	*Upper*	*Lower*	*Upper*	*Lower*	*Upper*
(20, 15)	0.234520	0.999956	0.173851	0.996843	0.146742	0.986182
(20, 18)	0.226190	0.999897	0.164349	0.994001	0.136197	0.975478
(50, 38)	0.233510	0.999906	0.172648	0.995105	0.145726	0.981074
(50, 45)	0.226566	0.999832	0.158817	0.989601	0.136007	0.970183
(100, 75)	0.234239	0.999890	0.173585	0.994800	0.146653	0.980353
(100, 90)	0.226585	0.999803	0.163905	0.991248	0.135979	0.968339

8. Conclusion

The findings of the simulation study are pretty interesting. The parameter has been under estimated for majority of the cases. The tendency of under estimation is more severe under mixture prior based on DLF and GELF. Similarly the increased true parametric values impose a negative impact on the convergence of the estimates. However, it can be observed that by increasing the sample size, the convergence of the estimated values toward the true parametric values tend to increase for each case. On the other hand, the amounts of posterior risks, based on each prior and loss function tend to decrease by increasing the sample size. It indicates that the estimators are consistent. It is interesting to note that the posterior risks under DLF and GELF based on inverse levy and gamma prior are independent of the choice of true parametric values. While in case of mixture prior for all loss functions and LLF under each prior, the amounts of posterior risks inflate for larger choice of true parametric values. Therefore, the extremely larger values of the parameters under these estimators may not be estimated with higher efficiency. However, the estimates under DLF and GELF based on inverse levy and gamma prior will be equally efficient for all choices of the parametric values. The bigger values of the mixing hyper-parameter of the mixture prior have a positive impact on the performance of the mixture prior.

The performance of the DLF and GELF is the best under inverse levy prior, while LLF provides top efficiency under mixture prior. Similarly, performance of inverse levy and gamma is superior under GELF, while the mixture prior works better under LLF. In case of LLF the improved results are observed for c = 1. And for GELF the smaller (absolute) choice of the shape parameter (p) of the loss function gives more precise estimates. The magnitude of risks under LLF and GELF are in close competition for smaller values of the parameter. However, when we increase the true parametric value the efficiency of GELF over LLF becomes evident. It is also observed that for the fixed n as r increases the performance of the concerned estimator becomes better in terms of the posterior risks in all the cases with an exception for linex loss function. Hence, it can be concluded that in order to estimate the smaller values of the shape parameter of the Kumaraswamy distribution the combination of LLF and mixture prior can be preferred. While in case of higher values of the parameter, the GELF under inverse levy prior can effectively be employed.

The limits for 95% posterior predictive intervals have been presented in tables 20-23. From these results, it can be observed that precision of the posterior predictive intervals is directly proportional to sample size and it is inversely proportional to true parametric values. The posterior predictions tend to be more specific under inverse levy prior.

It can also be assessed that under informative priors for a proficient choice of the parameters of the loss functions, the Bayesian inference has clear superiority over the frequentist one. The MLEs and UMVUE's are also under estimated but degree of underestimation is more in case of UMVUE. Furthermore, the relative efficiency of UMVUE is higher than that of MLE.

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