Hyperfuzzy Set and Hyperfuzzy Group

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Abstract

In this paper the concept of hyperfuzzy set is introduced and thereafter we define hyperfuzzy subgroup and a few of its properties are discussed. Also we define hyperfuzzy cosets and hyperfuzzy normal subgroup and study a few of their properties.

Keywords: *Fuzzy subgroup, Hyperfuzzy set, Hyperfuzzy subgroupoid, Hyperfuzzy subgroup, Hyperfuzzy coset, Hyperfuzzy normal subgroup*

1. Introduction

Most of the existing mathematical tools for formal modeling, reasoning and computing are crisp, deterministic and precise in character. But, in real life situation, the problem in economics, engineering, environment, social science, medical science etc. do not always involve crisp data. Consequently, we cannot successfully use the traditional classical methods because of various type of uncertainties presented in the problem. To deal with the uncertainties, fuzzy set theory can be considered as one of the mathematical tool. But, to deal with the indeterministic situation, we use deterministic approach through fuzzy set theory, because the membership function is always assumed to be single valued, which is exact one. Sometimes it is very difficult to determine the value of the membership function at a point in practical. It would be more logical and practical if the membership is assumed to be a multivalued function, that is, the image of the membership function at a point is an arbitrary subset of real numbers, in particular, an arbitrary subset of $[0,1]$. In fact, in this paper, we shall consider the membership function as a multivalued function, which will generalize the usual fuzzy set theory as well as interval-valued fuzzy set theory. In fact, for the consideration of multivalued membership function, the uncertain situation will be tackled by an uncertain manner.

The fundamental concept of fuzzy sets was initiated by Zadeh [5] in 1965 and opened a new path of thinking to mathematicians, engineers, physicists, chemists and many others due to its diverse applications in various fields. The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer science, control engineering, information science, coding theory, group theory, real analysis, measure theory etc. In 1971, Rosenfeld [4] first introduced the concept of fuzzy subgroups, which was the first fuzzification of any algebraic structure. Thereafter the notion of different fuzzy algebraic structures such as fuzzy ideals in rings and semirings etc. have seriously studied by many mathematicians. In 1975, Zadeh [6] introduced the concepts of interval-valued fuzzy sets (in short written by i-v fuzzy sets), where the values of the membership functions are intervals of real numbers instead of the real points. Thereafter in

1994, Biswas [1] define interval-valued fuzzy subgroups of the same nature of Rosenfeld's fuzzy subgroups and discuss some important results.

In the present paper we introduce the concepts of hyperfuzzy sets, where the values of the membership functions are assumed to be a subsets of $[0,1]$, which is a generalization of fuzzy sets and interval-valued fuzzy sets. In section 3, Hyperfuzzy subgroup is defined and a few of its basic properties are studied. In section 4, we define hyperfuzzy left and right cosets and also define hyperfuzzy normal subgroups and study a few of its properties.

2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel.

Definition 2.1. [5] Let X be a set. Then a mapping $\mu : X \to [0,1]$ is called a fuzzy subset of X.

Definition 2.2. [4] Let G be any group. A mapping $\mu : G \to [0,1]$ is called a fuzzy subgroup of G if

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ $\forall x, y \in G$,

(ii) $\mu(x^{-1}) \geq \mu(x)$ $\forall x \in G$.

Proposition 2.3. [4] If μ is a fuzzy subgroup of a group G having identity e, then:

(i) $\mu(x^{-1}) = \mu(x)$ $\forall x \in G$,

(ii) $\mu(e) \geq \mu(x)$ $\forall x \in G$.

Definition 2.4. [3] Let μ be a fuzzy subgroup of group G. Then for any $x \in G$, define the fuzzy right coset $\mu_{R(x)}$ and fuzzy left coset $\mu_{L(x)}$ by

$$\mu_{R(x)}(g) = \mu(gx^{-1}) \qquad \forall g \in G,$$
$$\mu_{L(x)}(g) = \mu(x^{-1}g) \qquad \forall g \in G.$$

Definition 2.5. [2] Let μ be a fuzzy subgroup of group G. Then μ is called a fuzzy normal subgroup if

$$\mu(xy) = \mu(yx) \qquad \forall x, y \in G.$$

Definition 2.6. [6] Let X be a set. An interval-valued fuzzy set (i.e. i-v fuzzy set) A defined on X is given by

$$A = \{(x, \bar{\mu}_A(x)) : x \in X\} \text{ where } \bar{\mu}_A : X \to D[0,1]$$

and $D[0,1]$ denoted the family of all closed subintervals of $[0,1]$ and $\bar{\mu}_A(x) = [\mu_A^L(x), \mu_A^R(x)]$ where μ_A^L , μ_A^L are two fuzzy subsets of X such that $\mu_A^L(x) \leq \mu_A^R(x)$, $\forall x \in X$.

Definition 2.7. [1] Consider two elements $D_1, D_2 \in D[0,1]$. If $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$, then $rmax(D_1, D_2) = [\max(a_1, a_2), \max(b_1, b_2)]$ and $rmin(D_1, D_2) = [\min(a_1, a_2), \min(b_1, b_2)]$.
Thus if $D_i = [a_i, b_i] \in D[0,1]$ for $i = 1, 2, 3, \dots\dots\dots\dots$, then we define $rsup_i D_i = [sup_i a_i, sup_i b_i]$ and $rinf_i D_i = [inf_i a_i, inf_i b_i]$. Now we call $D_1 \geq D_2$ iff $a_1 \geq a_2$ and $b_1 \geq b_2$. Similarly we may have $D_1 \leq D_2$ and $D_1 = D_2$. Clearly $D_1 \geq D_2 \nLeftrightarrow D_1 \supseteq D_2$.

Definition 2.8. [1] An i-v fuzzy subset $\bar{\mu}_G$ of a group X is called an i-v fuzzy subgroup if :
(i) $\bar{\mu}_G(xy) \geq rmin\{\bar{\mu}_G(x), \bar{\mu}_G(y)\}$ and
(ii) $\bar{\mu}_G(x^{-1}) \geq \bar{\mu}_G(x)$, where $x, y \in X$.

3. Hyperfuzzy subgroups

Now we define hyperfuzzy sets as a generalization of fuzzy sets [5] and interval-valued fuzzy sets [6]. Then we introduce the notion of hyperfuzzy subgroups and study some of its important properties.

Definition 3.1. Let X be a set. Then a mapping $\hat{\mu} : X \to P^*([0,1])$ is called a hyperfuzzy subset of X, where $P^*([0,1])$ denotes the set of all nonempty subset of $[0,1]$.

Definition 3.2. Let X be a nonempty set and $\hat{\mu}, \hat{\nu}$ be two hyperfuzzy subset of X. Then intersection of $\hat{\mu}$ and $\hat{\nu}$ is denoted by $\hat{\mu} \cap \hat{\nu}$ and defined by

$$(\hat{\mu} \cap \hat{\nu})(x) = \{\min\{p, q\} : p \in \hat{\mu}(x), q \in \hat{\nu}(x)\} \text{ for all } x \in X.$$

Union of $\hat{\mu}$ and $\hat{\nu}$ is denoted by $\hat{\mu} \cup \hat{\nu}$ and defined by

$$(\hat{\mu} \cup \hat{\nu})(x) = \{\max\{p, q\} : p \in \hat{\mu}(x), q \in \hat{\nu}(x)\} \text{ for all } x \in X.$$

Definition 3.3. Let X be a groupoid i.e. a set which is closed under a binary relation denoted multiplicatively. A mapping $\hat{\mu} : X \to P^*([0,1])$ is called a hyperfuzzy subgroupoid if $\forall x, y \in X$ following conditions hold:
(i) $\inf \hat{\mu}(xy) \geq \min\{\inf \hat{\mu}(x), \inf \hat{\mu}(y)\}$,
(ii) $\sup \hat{\mu}(xy) \geq \min\{\sup \hat{\mu}(x), \sup \hat{\mu}(y)\}$.

Definition 3.4. Let G be a group. A mapping $\hat{\mu} : G \to P^*([0,1])$ is called a hyperfuzzy subgroup of G if for all $x, y \in G$ following conditions hold:
(i) $\inf \hat{\mu}(xy) \geq \min\{\inf \hat{\mu}(x), \inf \hat{\mu}(y)\}$,
(ii) $\sup \hat{\mu}(xy) \geq \min\{\sup \hat{\mu}(x), \sup \hat{\mu}(y)\}$,
(iii) $\inf \hat{\mu}(x^{-1}) \geq \inf \hat{\mu}(x)$,
(iv) $\sup \hat{\mu}(x^{-1}) \geq \sup \hat{\mu}(x)$.

Note 3.5. In definition 3.4, if $\hat{\mu} : G \to [0,1]$ then $\hat{\mu}(x), \forall x \in G$ are real points in $[0,1]$ and also $\inf \hat{\mu}(x) = \sup \hat{\mu}(x) = \hat{\mu}(x), \forall x \in G$. Thus definition 3.4 reduces to definition 2.2 of Rosenfeld's fuzzy subgroups. Also if $\hat{\mu} : G \to D([0,1])$ then $\hat{\mu}(x), \forall x \in G$ are closed subintervals of $[0,1]$ and conditions (i) , (ii) of definition 3.4 together implies the condition (i) of definition 2.8. Similarly conditions (iii) , (iv) of definition 3.4 together implies the condition (ii) of definition 2.8. Thus definition 3.4 reduces to definition 2.8 of interval-valued fuzzy subgroups.
So, hyperfuzzy subgroups is a generalization of fuzzy subgroups and interval-valued fuzzy subgroups.

Proposition 3.6. If $\hat{\mu}$ is a hyperfuzzy subgroupoid of a finite group G, then $\hat{\mu}$ is a hyperfuzzy subgroup of G.

Proof. Let $x \in G$. Since G is finite, x has finite order, say n. Then $x^n = e$, where e is the identity of G. Thus $x^{-1} = x^{n-1}$. Now using the definition of hyperfuzzy subgroupoid, we have
$\inf \hat{\mu}(x^{-1}) = \inf \hat{\mu}(x^{n-1}) = \inf \hat{\mu}(x^{n-2}x) \geq \min\{\inf \hat{\mu}(x^{n-2}), \inf \hat{\mu}(x)\}$.
Again, $\inf \hat{\mu}(x^{n-2}) = \inf \hat{\mu}(x^{n-3}x) \geq \min\{\inf \hat{\mu}(x^{n-3}), \inf \hat{\mu}(x)\}$.
Then we have $\inf \hat{\mu}(x^{-1}) \geq \min\{\inf \hat{\mu}(x^{n-3}), \inf \hat{\mu}(x)\}$.
So applying definition of hyperfuzzy subgroupoid repeatedly, we have that
$\inf \hat{\mu}(x^{-1}) \geq \inf \hat{\mu}(x)$.
Similarly we have
$\sup \hat{\mu}(x^{-1}) \geq \sup \hat{\mu}(x)$.
Hence $\hat{\mu}$ is a hyperfuzzy subgroup of G.

Proposition 3.7. If $\hat{\mu}$ is a hyperfuzzy subgroup of a group G having the identity e, then for all $x \in X$,
(i) $\inf \hat{\mu}(x^{-1}) = \inf \hat{\mu}(x)$ and $\sup \hat{\mu}(x^{-1}) = \sup \hat{\mu}(x)$.
(ii) $\inf \hat{\mu}(e) \geq \inf \hat{\mu}(x)$ and $\sup \hat{\mu}(e) \geq \sup \hat{\mu}(x)$.

Proof. (i) As $\hat{\mu}$ is a hyperfuzzy subgroup of group G, then
$\inf \hat{\mu}(x^{-1}) \geq \inf \hat{\mu}(x) \quad \forall x \in G$.
Again $\inf \hat{\mu}(x) = \inf \hat{\mu}((x^{-1})^{-1}) \geq \inf \hat{\mu}(x^{-1})$.
So $\inf \hat{\mu}(x^{-1}) = \inf \hat{\mu}(x)$.
Similarly we can prove that $\sup \hat{\mu}(x^{-1}) = \sup \hat{\mu}(x)$.
(ii) $\inf \hat{\mu}(e) = \inf \hat{\mu}(xx^{-1}) \geq \min\{\inf \hat{\mu}(x), \inf \hat{\mu}(x^{-1})\} = \inf \hat{\mu}(x)$ and
$\sup \hat{\mu}(e) = \sup \hat{\mu}(xx^{-1}) \geq \min\{\sup \hat{\mu}(x), \sup \hat{\mu}(x^{-1})\} = \sup \hat{\mu}(x)$.
Hence the proposition is proved.

Proposition 3.8. A hyperfuzzy subset $\hat{\mu}$ of group G is a hyperfuzzy subgroup iff for all $x, y \in G$ followings are hold:
(i) $\inf \hat{\mu}(xy^{-1}) \geq \min\{\inf \hat{\mu}(x), \inf \hat{\mu}(y)\}$,
(ii) $\sup \hat{\mu}(xy^{-1}) \geq \min\{\sup \hat{\mu}(x), \sup \hat{\mu}(y)\}$.

Proof. At first let $\hat{\mu}$ be a hyperfuzzy subgroup of G and $x, y \in G$. Then

$\inf \hat{\mu}(xy^{-1}) \geq \min\{\inf \hat{\mu}(x), \inf \hat{\mu}(y^{-1})\} = \min\{\inf \hat{\mu}(x), \inf \hat{\mu}(y)\}$ and
$\sup \hat{\mu}(xy^{-1}) \geq \min\{\sup \hat{\mu}(x), \sup \hat{\mu}(y^{-1})\} = \min\{\sup \hat{\mu}(x), \sup \hat{\mu}(y)\}$.
Conversely, let $\hat{\mu}$ be a hyperfuzzy subset of G and given conditions hold.
Then for all $x \in G$, we have

$$\inf \hat{\mu}(e) = \inf \hat{\mu}(xx^{-1}) \geq \min\{\inf \hat{\mu}(x), \inf \hat{\mu}(x)\} = \inf \hat{\mu}(x) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\sup \hat{\mu}(e) = \sup \hat{\mu}(xx^{-1}) \geq \min\{\sup \hat{\mu}(x), \sup \hat{\mu}(x)\} = \sup \hat{\mu}(x) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

So, $\inf \hat{\mu}(x^{-1}) = \inf \hat{\mu}(ex^{-1}) \geq \min\{\inf \hat{\mu}(e), \inf \hat{\mu}(x)\} = \inf \hat{\mu}(x)$, by (1)
and $\sup \hat{\mu}(x^{-1}) = \sup \hat{\mu}(ex^{-1}) \geq \min\{\sup \hat{\mu}(e), \sup \hat{\mu}(x)\} = \sup \hat{\mu}(x)$, by (2)
Again, $\inf \hat{\mu}(xy) \geq \min\{\inf \hat{\mu}(x), \inf \hat{\mu}(y^{-1})\}$, using given codition
$\geq \min\{\inf \hat{\mu}(x), \inf \hat{\mu}(y)\}$, by above
And $\sup \hat{\mu}(xy) \geq \min\{\sup \hat{\mu}(x), \sup \hat{\mu}(y^{-1})\}$, using given codition
$\geq \min\{\sup \hat{\mu}(x), \sup \hat{\mu}(y)\}$, by above
Hence $\hat{\mu}$ is a hyperfuzzy subgroup of G. Hence the proposition is proved.

Proposition 3.9. Intersection of any two hyperfuzzy subgroups of a group G is also a hyperfuzzy subgroup of G.

Proof. Let $\hat{\mu}$ and $\hat{\nu}$ be any two hyperfuzzy subgroup of G and $x, y \in G$. Then
$\inf(\hat{\mu} \cap \hat{\nu})(xy^{-1}) = \min\{\inf \hat{\mu}(xy^{-1}), \inf \hat{\nu}(xy^{-1})\}$, by definition 3.2
$\geq \min\{\min\{\inf \hat{\mu}(x), \inf \hat{\mu}(y)\}, \min\{\inf \hat{\nu}(x), \inf \hat{\nu}(y)\}\}$, by prop. 3.8
$= \min\{\min\{\inf \hat{\mu}(x), \inf \hat{\nu}(x)\}, \min\{\inf \hat{\mu}(y), \inf \hat{\nu}(y)\}\}$
$= \min\{\inf(\hat{\mu} \cap \hat{\nu})(x), \inf(\hat{\mu} \cap \hat{\nu})(y)\} \quad \ldots \quad \ldots \quad \ldots \quad (3)$
Again,
$\sup(\hat{\mu} \cap \hat{\nu})(xy^{-1}) = \min\{\sup \hat{\mu}(xy^{-1}), \sup \hat{\nu}(xy^{-1})\}$, by definition 3.2
$\geq \min\{\min\{\sup \hat{\mu}(x), \sup \hat{\mu}(y)\}, \min\{\sup \hat{\nu}(x), \sup \hat{\nu}(y)\}\}$, by prop. 3.8
$= \min\{\min\{\sup \hat{\mu}(x), \sup \hat{\nu}(x)\}, \min\{\sup \hat{\mu}(y), \sup \hat{\nu}(y)\}\}$
$= \min\{\sup(\hat{\mu} \cap \hat{\nu})(x), \sup(\hat{\mu} \cap \hat{\nu})(y)\} \quad \ldots \quad \ldots \quad \ldots \quad (4)$
Hence by (3) and (4) and using proposition 3.8, we say that $\hat{\mu} \cap \hat{\nu}$ is a hyperfuzzy subgroup of G. Hence the proposition is proved.

Note 3.10. Proposition 3.9 is also true for arbitrary intersection of hyperfuzzy subgroups.

4. Hyperfuzzy Cosets and Normal Subgroups

Given a fuzzy subgroup μ of group G, one could define the fuzzy left cosets and the fuzzy right cosets of G relative to μ. We now define hyperfuzzy left cosets and hyperfuzzy right cosets analogously.

Definition 4.1. Let $\hat{\mu}$ be a hyperfuzzy subgroup of a group G. For any $x \in G$, define a mapping $\hat{\mu}_{L(x)} : G \to P^*([0,1])$ by

$$\hat{\mu}_{L(x)}(g) = \hat{\mu}(x^{-1}g) \quad \forall\, g \in G,$$

And also define a mapping $\hat{\mu}_{R(x)} : G \to P^*([0,1])$ by

$\hat{\mu}_{R(x)}(g) = \hat{\mu}(gx^{-1}) \ \ \forall g \in G$.

Then $\hat{\mu}_{L(x)}$, $\hat{\mu}_{R(x)}$ are respectively called the hyperfuzzy left coset and hyperfuzzy right coset of group G determined by x and $\hat{\mu}$.

In crisp concept, a subgroup H of a group G for which $aH = Ha$ holds for all $a \in G$ i.e. left coset equals to corresponding right coset, is called normal subgroup of G. Here we extend this concepts for hyperfuzzy set. A hyperfuzzy subgroup $\hat{\mu}$ of a group G is called normal if $\forall x, g \in G$,

$\inf \hat{\mu}_{L(x)}(g) = \inf \hat{\mu}_{R(x)}(g)$ and $\sup \hat{\mu}_{L(x)}(g) = \sup \hat{\mu}_{R(x)}(g)$ i.e.

$\inf \hat{\mu}(x^{-1}g) = \inf \hat{\mu}(gx^{-1})$ and $\sup \hat{\mu}(x^{-1}g) = \sup \hat{\mu}(gx^{-1})$.

So, we give formal definition of hyperfuzzy normal subgroup as follows:

Definition 4.2. Let $\hat{\mu}$ be a hyperfuzzy subgroup of a group G. Then $\hat{\mu}$ is called a hyperfuzzy normal subgroup of G if

$$\inf \hat{\mu}(xy) = \inf \hat{\mu}(yx) \text{ and } \sup \hat{\mu}(xy) = \sup \hat{\mu}(yx) \quad \forall x, y \in G.$$

Proposition 4.3. The intersection of any two hyperfuzzy normal subgroups of a group G is also a hyperfuzzy normal subgroup of G.

Proof. Let $\hat{\mu}, \hat{\nu}$ be two hyperfuzzy normal subgroups of group G. By proposition 3.9. $\hat{\mu} \cap \hat{\nu}$ is a hyperfuzzy subgroup of G. Let $x, y \in G$ then by definition 3.2,

$\inf(\hat{\mu} \cap \hat{\nu})(xy) = \min\{\inf \hat{\mu}(xy), \inf \hat{\nu}(xy)\}$

$= \min\{\inf \hat{\mu}(yx), \inf \hat{\nu}(yx)\}$, by definition 4.2

$= \inf(\hat{\mu} \cap \hat{\nu})(yx)$.

Similarly, $\sup(\hat{\mu} \cap \hat{\nu})(xy) = \sup(\hat{\mu} \cap \hat{\nu})(yx)$.

This shows that $\hat{\mu} \cap \hat{\nu}$ is a hyperfuzzy normal subgroup of G. Hence, the proposition is proved.

Note 4.4. The intersection of any arbitrary collection of hyperfuzzy normal subgroups of a group G is also a hyperfuzzy normal subgroup of G.

Theorem 4.5. Let $\hat{\mu}$ be a hyperfuzzy subgroup of a group G and $a \in G$. Then the hyperfuzzy subset $\hat{\nu} : G \to P^*([0,1])$ defined by $\hat{\nu}(x) = \hat{\mu}(a^{-1}xa) \ \forall x \in G$, is a hyperfuzzy subgroup of G.

Proof. Let $x, y \in G$. Then for all $a \in G$,

$\inf \hat{\nu}(xy^{-1}) = \inf \hat{\mu}(a^{-1}xy^{-1}a)$, by definition of $\hat{\nu}$

$= \inf \hat{\mu}(a^{-1}xaa^{-1}y^{-1}a)$

$= \inf \hat{\mu}((a^{-1}xa)(a^{-1}ya)^{-1})$

$\geq \min\{\inf \hat{\mu}(a^{-1}xa), \inf \hat{\mu}(a^{-1}ya)\}$, since $\hat{\mu}$ is a hyperfuzzy subgroup

$= \min\{\inf \hat{\nu}(x), \inf \hat{\nu}(y)\}$.

Again, $\sup \hat{\nu}(xy^{-1}) = \sup \hat{\mu}(a^{-1}xy^{-1}a)$, by definition of $\hat{\nu}$

$= \sup \hat{\mu}(a^{-1}xaa^{-1}y^{-1}a)$

$= \sup \hat{\mu}((a^{-1}xa)(a^{-1}ya)^{-1})$

$\geq \min\{\sup\hat{\mu}(a^{-1}xa), \sup\hat{\mu}(a^{-1}ya)\}$, since $\hat{\mu}$ is a hyperfuzzy subgroup
$= \min\{\sup\hat{\nu}(x), \sup\hat{\nu}(y)\}$.
Hence by proposition 3.8, $\hat{\nu}$ is a hyperfuzzy subgroup of G.

Definition 4.6. Let $\hat{\mu}$ and $\hat{\nu}$ be two hyperfuzzy subgroups of a group G. We say that $\hat{\nu}$ is conjugate to $\hat{\mu}$ if for some $a \in G$ we have that

$$\inf\hat{\nu}(x) = \inf\hat{\mu}(a^{-1}xa) \text{ and } \sup\hat{\nu}(x) = \sup\hat{\mu}(a^{-1}xa) \quad \forall x \in G.$$

Proposition 4.7. For any hyperfuzzy subset $\hat{\mu}$ of a group G and for all $x, y \in G$, followings are equivalent:
(i) $\inf\hat{\mu}(xyx^{-1}) = \inf\hat{\mu}(y)$ and $\sup\hat{\mu}(xyx^{-1}) = \sup\hat{\mu}(y)$.
(ii) $\inf\hat{\mu}(xy) = \inf\hat{\mu}(yx)$ and $\sup\hat{\mu}(xy) = \sup\hat{\mu}(yx)$.
(iii) $\inf\hat{\mu}_{L(x)}(y) = \inf\hat{\mu}_{R(x)}(y)$ and $\sup\hat{\mu}_{L(x)}(y) = \sup\hat{\mu}_{R(x)}(y)$.

Proof. Let $x, y \in G$ and $\hat{\mu}$ be a hyperfuzzy subset of group G.
$(i) \Rightarrow (ii)$: $\inf\hat{\mu}(yx) = \inf\hat{\mu}(x^{-1}xyx) = \inf\hat{\mu}(xy)$, using (i)
and $\sup\hat{\mu}(yx) = \sup\hat{\mu}(x^{-1}xyx) = \sup\hat{\mu}(xy)$.
$(ii) \Rightarrow (iii)$: $\inf\hat{\mu}_{L(x)}(y) = \inf\hat{\mu}(x^{-1}y) = \inf\hat{\mu}(yx^{-1})$, using (ii)
$= \inf\hat{\mu}_{R(x)}(y)$.
and $\sup\hat{\mu}_{L(x)}(y) = \sup\hat{\mu}(x^{-1}y) = \sup\hat{\mu}(yx^{-1}) = \sup\hat{\mu}_{R(x)}(y)$.
$(iii) \Rightarrow (i)$: $\inf\hat{\mu}(xyx^{-1}) = \inf\hat{\mu}_{R(x)}(xy) = \inf\hat{\mu}_{L(x)}(xy)$, using (iii)
$= \inf\hat{\mu}(x^{-1}xy) = \inf\hat{\mu}(y)$.
and
$\sup\hat{\mu}(xyx^{-1}) = \sup\hat{\mu}_{R(x)}(xy) = \sup\hat{\mu}_{L(x)}(xy) = \sup\hat{\mu}(x^{-1}xy) = \sup\hat{\mu}(y)$.
Hence the proposition is proved.

Definition 4.8. A hyperfuzzy subgroup $\hat{\mu}$ of a group G is called self conjugate hyperfuzzy subgroup if for all $a, x \in G$, we have that

$$\inf\hat{\mu}(x) = \inf\hat{\mu}(a^{-1}xa) \text{ and } \sup\hat{\mu}(x) = \sup\hat{\mu}(a^{-1}xa).$$

Theorem 4.9. A hyperfuzzy subgroup $\hat{\mu}$ of a group G is normal iff $\hat{\mu}$ is self conjugate hyperfuzzy subgroup.

Proof. Let $\hat{\mu}$ be a hyperfuzzy normal subgroup of group G. Then

$$\inf\hat{\mu}(xy) = \inf\hat{\mu}(yx) \text{ and } \sup\hat{\mu}(xy) = \sup\hat{\mu}(yx) \quad \forall x, y \in G.$$

Then by using proposition 4.7 we have

$$\inf\hat{\mu}(xyx^{-1}) = \inf\hat{\mu}(y) \text{ and } \sup\hat{\mu}(xyx^{-1}) = \sup\hat{\mu}(y) \quad \forall x, y \in G.$$

So, $\hat{\mu}$ is a self conjugate hyperfuzzy subgroup.
Conversely, let $\hat{\mu}$ be a self conjugate hyperfuzzy subgroup.
Then $\inf\hat{\mu}(xyx^{-1}) = \inf\hat{\mu}(y)$ and $\sup\hat{\mu}(xyx^{-1}) = \sup\hat{\mu}(y) \quad \forall x, y \in G$.
Then again by proposition 4.7 we have

$$\inf\hat{\mu}(xy) = \inf\hat{\mu}(yx) \text{ and } \sup\hat{\mu}(xy) = \sup\hat{\mu}(yx) \quad \forall x, y \in G.$$

So, $\hat{\mu}$ is a hyperfuzzy normal subgroup. This completes the proof.

Definition 4.10. Let $\hat{\mu}$ be a hyperfuzzy subgroup of group G. Then normalizer of $\hat{\mu}$ is defined by

$N(\hat{\mu}) = \{a \in G : \forall x \in G, \inf\hat{\mu}(a^{-1}xa) = \inf\hat{\mu}(x), \sup\hat{\mu}(a^{-1}xa) = \sup\hat{\mu}(x)\}$.

Theorem 4.11. Let $\hat{\mu}$ be a hyperfuzzy normal subgroup of group G. Then
(i) $N(\hat{\mu})$ is a subgroup of G.
(ii) $\hat{\nu} : N(\hat{\mu}) \rightarrow P^*([0,1])$ is defined by $\hat{\nu}(x) = \hat{\mu}(x)\ \forall x \in N(\hat{\mu})$.
Then $\hat{\nu}$ is a hyperfuzzy normal subgroup of $N(\hat{\mu})$.

Proof. (i) Let $x, y \in N(\hat{\mu})$. Then for all $g \in G$,
$\inf\hat{\mu}((xy)^{-1}g(xy)) = \inf\hat{\mu}(y^{-1}x^{-1}gxy) = \inf\hat{\mu}(x^{-1}gx)$, since
$y \in N(\hat{\mu})$, $x^{-1}gx \in G$
$= \inf\hat{\mu}(g)$, since $x \in N(\hat{\mu})$.
Similarly, $\sup\hat{\mu}((xy)^{-1}g(xy)) = \sup\hat{\mu}(g)$. So, $xy \in N(\hat{\mu})$.
Again $g \in G$, $x \in N(\hat{\mu}) \Rightarrow xgx^{-1} \in G$. Then for all $g \in G$,
$\inf\hat{\mu}(xgx^{-1}) = \inf\hat{\mu}(x^{-1}(xgx^{-1})x)$, since $x \in N(\hat{\mu})$, $xgx^{-1} \in G$
$= \inf\hat{\mu}(x^{-1}xgx^{-1}x) = \inf\hat{\mu}(g)$.
Similarly, $\sup\hat{\mu}(xgx^{-1}) = \sup\hat{\mu}(g)$.
So, $x^{-1} \in N(\hat{\mu})$. Hence $N(\hat{\mu})$ is a subgroup of G.

(ii) Since $\hat{\mu}$ is a hyperfuzzy subgroup of G and we have proved that $N(\hat{\mu})$ is a subgroup of G. Then $\hat{\mu}$ is a hyperfuzzy subgroup of $N(\hat{\mu})$. Hence $\hat{\nu}$ is a hyperfuzzy subgroup of $N(\hat{\mu})$. Now we have to prove $\hat{\nu}$ is normal.
Since $N(\hat{\mu})$ is a subgroup of G, then $x, y \in N(\hat{\mu}) \Rightarrow x^{-1}yx \in N(\hat{\mu})$.
Now by definition of $\hat{\nu}$, we have for all $x, y \in N(\hat{\mu})$,
$\inf\hat{\nu}(x^{-1}yx) = \inf\hat{\mu}(x^{-1}yx)$, since $x^{-1}yx \in N(\hat{\mu})$
$= \inf\hat{\mu}(y)$, since $x \in N(\hat{\mu})$
$= \inf\hat{\nu}(y)$, since $y \in N(\hat{\mu})$.
Similarly, $\sup\hat{\nu}(x^{-1}yx) = \sup\hat{\nu}(y)$.
Hence $\hat{\nu}$ is self conjugate hyperfuzzy subgroup of $N(\hat{\mu})$.
Hence by theorem 4.9, $\hat{\nu}$ is a hyperfuzzy normal subgroup of $N(\hat{\mu})$.
This completes the proof.

Theorem 4.12. Let $\hat{\mu}$ be a hyperfuzzy subgroup of group G. Then $\hat{\mu}$ is a hyperfuzzy normal subgroup of $G \Leftrightarrow N(\hat{\mu}) = G$.

Proof. This theorem follows from Theorem 4.9 and Definitions 4.8, 4.10.

Remark 4.13. If $\hat{\mu}$ is a hyperfuzzy subgroup of G but not normal then $N(\hat{\mu})$ is a proper subgroup of G by Theorem 4.11 and Theorem 4.12.

Theorem 4.14. Let $\hat{\mu}$ be a hyperfuzzy subgroup of a group G. Define

$H = \{g \in G : \inf\hat{\mu}(g) = \inf\hat{\mu}(e)\ \ and\ \ \sup\hat{\mu}(g) = \sup\hat{\mu}(e)\}$.

$K = \{ g \in G : \forall x \in G, \ \inf \hat{\mu}_{R(g)}(x) = \inf \hat{\mu}_{R(e)}(x) \ \ and \ \ \sup \hat{\mu}_{R(g)}(x) = \sup \hat{\mu}_{R(e)}(x) \}.$

Proof. Let $\hat{\mu}$ be a hyperfuzzy subgroup of a group G and $g, h \in H$. Then by proposition 3.8 we have that $\inf \hat{\mu}(gh^{-1}) \geq \min\{ \inf \hat{\mu}(g), \inf \hat{\mu}(h) \}$
$= \min\{ \inf \hat{\mu}(e), \inf \hat{\mu}(e) \}$, since $g, h \in H$
$= \inf \hat{\mu}(e)$.
Again by proposition 3.7, we have $\inf \hat{\mu}(e) \geq \inf \hat{\mu}(gh^{-1})$.
Hence $\inf \hat{\mu}(gh^{-1}) = \inf \hat{\mu}(e)$.
Similarly, by proposition 3.8 and 3.7, we have $\sup \hat{\mu}(gh^{-1}) = \sup \hat{\mu}(e)$.
So, $gh^{-1} \in H$. Hence H is a subgroup of group G.
We now show that $H = K$. Let $h \in H$.
So $\inf \hat{\mu}(h) = \inf \hat{\mu}(e) \ \ and \ \ \sup \hat{\mu}(h) = \sup \hat{\mu}(e)$.
Now for all $x \in G$, $\inf \hat{\mu}_{R(h)}(x) = \inf \hat{\mu}(xh^{-1})$
$\geq \min\{ \inf \hat{\mu}(x), \inf \hat{\mu}(h) \}$, since $\hat{\mu}$ is a hyperfuzzy subgroup
$= \min\{ \inf \hat{\mu}(x), \inf \hat{\mu}(e) \}$, since $h \in H$
$= \inf \hat{\mu}(x)$, since by proposition 3.7, $\inf \hat{\mu}(e) \geq \inf \hat{\mu}(x)$
$= \inf \hat{\mu}(xe^{-1}) = \inf \hat{\mu}_{R(e)}(x)$ ... ... (5)
Again, $\inf \hat{\mu}_{R(e)}(x) = \inf \hat{\mu}(xe^{-1})$
$= \inf \hat{\mu}(x) = \inf \hat{\mu}(xh^{-1}h)$, where $h \in H$
$\geq \min\{ \inf \hat{\mu}(xh^{-1}), \inf \hat{\mu}(h) \}$, since $\hat{\mu}$ is a hyperfuzzy subgroup
$= \min\{ \inf \hat{\mu}(xh^{-1}), \inf \hat{\mu}(e) \}$, since $h \in H$
$= \inf \hat{\mu}(xh^{-1})$, since by proposition 3.7, $\inf \hat{\mu}(e) \geq \inf \hat{\mu}(xh^{-1})$
$= \inf \hat{\mu}_{R(h)}(x)$ ... ... (6)
Hence by (5) and (6) we have $\inf \hat{\mu}_{R(h)}(x) = \inf \hat{\mu}_{R(e)}(x)$.
Similarly, we can prove that $\sup \hat{\mu}_{R(h)}(x) = \sup \hat{\mu}_{R(e)}(x)$.
This implies $h \in K$. Hence $H \subseteq K$.
Now suppose $k \in K$. Then for all $x \in G$, we have that
$\inf \hat{\mu}_{R(k)}(x) = \inf \hat{\mu}_{R(e)}(x) \ \ and \ \ \sup \hat{\mu}_{R(k)}(x) = \sup \hat{\mu}_{R(e)}(x)$.
This implies $\inf \hat{\mu}(xk^{-1}) = \inf \hat{\mu}(x)$ and $\sup \hat{\mu}(xk^{-1}) = \sup \hat{\mu}(x)$.
Choosing $x = e$, we obtain
$\inf \hat{\mu}(k^{-1}) = \inf \hat{\mu}(e)$ and $\sup \hat{\mu}(k^{-1}) = \sup \hat{\mu}(e)$.
Hence $k^{-1} \in H$. Since H is a subgroup, so $k \in H$.
Thus we have $K \subseteq H$. Therefore $H = K$ and K is also a subgroup of G. This completes the proof.

Corollary 4.15. With the same notation as in Theorem 4.14, if $\hat{\mu}$ is a hyperfuzzy normal subgroup of a group G then H is a normal subgroup of G.

Proof. Let $a \in G$ and $y \in H$. Then

$\inf \hat{\mu}(a^{-1}ya) = \inf \hat{\mu}(y)$, since $\hat{\mu}$ is hyperfuzzy normal subgroup of G
$= \inf \hat{\mu}(e)$, since $y \in H$.
Similarly, $\sup \hat{\mu}(a^{-1}ya) = \sup \hat{\mu}(y) = \sup \hat{\mu}(e)$.
So, $a^{-1}ya \in H$. Hence H is a normal subgroup of G.

Note 4.16. In Theorem 4.14, if
$K = \{ g \in G : \forall x \in G, \inf \hat{\mu}_{L(g)}(x) = \inf \hat{\mu}_{L(e)}(x) \ and \ \sup \hat{\mu}_{L(g)}(x) = \sup \hat{\mu}_{L(e)}(x) \}$
then theorem also follows.

Theorem 4.17. Let $\hat{\mu}$ be a hyperfuzzy subgroup of group G and $\hat{\nu}$ be a hyperfuzzy normal subgroup of G. Then $\hat{\mu} \cap \hat{\nu}$ is a hyperfuzzy normal subgroup of the group

$$H = \{ g \in G : \inf \hat{\mu}(g) = \inf \hat{\mu}(e) \ and \ \sup \hat{\mu}(g) = \sup \hat{\mu}(e) \}.$$

Proof. By Theorem 4.14, H is a subgroup of G and by proposition 3.9, $\hat{\mu} \cap \hat{\nu}$ is a hyperfuzzy subgroup of G. So, $\hat{\mu} \cap \hat{\nu}$ is a hyperfuzzy subgroup of H. We now show that $\hat{\mu} \cap \hat{\nu}$ is a hyperfuzzy normal subgroup of H. Let $x, y \in H$. Since H is a subgroup, then $xy, yx \in H$.
Now $\inf(\hat{\mu} \cap \hat{\nu})(xy) = \min\{ \inf \hat{\mu}(xy), \inf \hat{\nu}(xy) \}$
$= \min\{ \inf \hat{\mu}(xy), \inf \hat{\nu}(yx) \}$, since $\hat{\nu}$ is hyperfuzzy normal subgroup
$= \min\{ \inf \hat{\mu}(yx), \inf \hat{\nu}(yx) \}$, since $xy \in H \Rightarrow \inf \hat{\mu}(xy) = \inf \hat{\mu}(e) = \inf \hat{\mu}(yx)$
$= \inf(\hat{\mu} \cap \hat{\nu})(yx)$.
Similarly, we can prove that $\sup(\hat{\mu} \cap \hat{\nu})(xy) = \sup(\hat{\mu} \cap \hat{\nu})(yx)$.
Hence $\hat{\mu} \cap \hat{\nu}$ is a hyperfuzzy normal subgroup of H. This completes the proof.

Acknowledgement

The authors are grateful to SERSC Korea Branch for their cooperation with financial support for the articles which are published in IJAST and also grateful to the referees for their comments and suggestions.

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