An Analysis of Approximate Equalities based on Rough Set Theory

B. K. Tripathy
Senior Professor, SCS, VIT University, Vellore – 632014, Tamilnadu, India
tripathybk@rediffmail.com

Abstract

Three types of rough equalities were introduced by Novotny and Pawlak ([1, 2, 3]), which take care of approximate equalities of sets. These sets may not be equal in the usual sense. These notions were generalized by Tripathy, Mitra and Ojha ([10]), who introduced the concepts of rough equivalences of sets. These approximate equalities of sets capture equality of the concerned sets at a higher level than their corresponding rough equalities. Some more properties were proved in [11]. In this paper, we introduce two other types of approximate equalities of sets, called the approximate rough equivalences and approximate rough equality. We study some properties of these four types of approximate equalities and analyse their relevance from the application point of view.

Keywords: *Bottom R-equal, top R-equal, R-equal, bottom R-equivalent, top R-equivalent, R-equivalent, approximate R-equivalent*

1. Introduction

The observation that most of our traditional tools for formal modeling, reasoning and computing are crisp, deterministic and precise in character, which restricts their applicability in real life situations, led to the extension of the concept of crisp sets so as to model imprecise data and enhance their modeling power. One such approach to capture impreciseness is the notion of Rough Sets, introduced by Pawlak [4]. The basic assumption of rough set theory is that human knowledge about a universe depends upon their capability to classify its objects. Classifications of a universe and equivalence relations defined on it are known to be interchangeable notions. So, for mathematical reasons equivalence relations were considered by Pawlak to define rough sets. A rough set of a set X is represented by a pair of crisp sets, called the lower approximation of X comprising of elements which belong to X definitely and upper approximation of X comprising of elements which are possibly in X with respect to the available information.

The equality of two sets in set theory deals with the elements they comprise of. This notion is independent of the user or more precisely the user knowledge about the universe dealt with. An attempt to incorporate the user knowledge about the structure of the universe in concluding about the equality of two sets was made by Novotny and Pawlak ([1, 2, 3]), which led to the introduction of the concepts of rough or approximate equality. This is an important feature from the application point of view. The reason being that, in certain cases it might not be possible for us to conclude about the equality of two sets from the available knowledge in the mathematical sense. But, we can only say that, according to our state of knowledge, they have close features which are enough to assume that they are approximately equal. That is, basing upon our knowledge and requirement we can assume that the two sets are indistinguishable.

Even these definitions of approximate equalities were found to have limited scope in defining approximate equality of sets and the concepts of rough equivalences were introduced and studied by Tripathy et al. ([10]). Their notions capture the approximate equalities of two sets at a higher level and obviously are more general. Some examples from real life were taken in [10] and [11] to illustrate the superiority of the new notions of rough equivalences over the existing notions of approximate equalities.

Several properties of approximate equalities were established by Novotny and Pawlak [1, 2, 3]. These properties were stated to be false when the notion of lower approximate equality is replaced with upper approximate equality and vice versa. However, it is shown in [10] that this is not true in case of some of the properties and in some other cases it is true when additional conditions are assumed. In [10] attempts were made to extend these properties to the generalized situation of rough equivalences. It was found that the properties failed to hold in their full generality and mostly parts were found to hold true. The other parts were established under suitable conditions. The validity of some basic algebraic properties involving union, intersection and complementation of sets were tested for their validity with equality of sets being replaced by rough equivalence in [11].

In this paper, we introduce two more types of approximate equalities of sets and make a comparative study of these four types of approximate equalities of sets. We shall show that the notion of approximate equivalence, which is the third kind, seems to be the most natural among these four. The fourth type is also to be discussed. To establish our claim, we shall use two examples; namely that dealing with the universe of cattle in a locality and the universe of shares in a country are to be taken for consideration.

1.1 Basic Rough Sets

Let *U* be a universe of discourse and *R* be an equivalence relation over *U*. By *U/R* we denote the family of all equivalence class of *R*, referred to as categories or concepts of *R* and the equivalence class of an element $x \in U$, is denoted by $[x]_R$. By a knowledge base, we understand a relational system *K* = (*U*, *R*), where *U* is as above and *R* is a family of equivalence relations over *U*. For any subset $P\ (\neq \phi) \subseteq R$, the intersection of all equivalence relations in *P* is denoted by IND (P) and is called the indiscernibility relation over *P*. Given any $X \subseteq U$ and $R \in$ IND (K), we associate two subsets, $\underline{R}X = \cup\{Y \in U/R : Y \subseteq X\}$ and $\overline{R}X = \cup\{Y \in U/R : Y \cap X \neq \phi\}$, called the R-lower and R-upper approximations of X respectively.

The R-boundary of X is denoted by BNR (X) and is given by $BN_R(X) = \overline{R}X - \underline{R}X$. The elements of $\underline{R}X$ are those elements of *U*, which can certainly be classified as elements of X, and the elements of $\overline{R}X$ are those elements of U, which can possibly be classified as elements of X, employing knowledge of *R*. We say that *X* is rough with respect to *R* if and only if $\underline{R}X \neq \overline{R}X$, equivalently $BN_R(X) \neq \phi$. *X* is said to be R-definable if and only if $\underline{R}X = \overline{R}X$, or $BN_R(X) = \phi$.

2. Approximate Equalities of Sets

As described in the introduction, sometimes exact equality (equality in the mathematical sense) is too stringent to apply in day to day life. We often talk about equality of sets or domains, which can be considered to be equal for the purpose or under the circumstances in real life situations. So, approximate equalities play a significant role in our reasoning. Also, it is dependent upon the knowledge the assessors have about the domain under consideration as a whole but mostly not the knowledge about individuals. The question now arises so as to find metrics to measure the equality. This is provided by the equivalence relations defined over the domain. Of course, this can be extended to other types of relations also (see for instance [12, 13]). For the time being we shall focus our attention upon equivalence relations only in this paper.

2.1. The kinds of Approximate Equalities

In order to define the approximate equalities we take the following conditions on the lower and upper approximations of sets X and Y:

(L1) $\underline{R}X = \underline{R}Y$

(L2) both $\underline{R}X$ and $\underline{R}Y$ are ϕ or not ϕ together

(U1) $\bar{R}X = \bar{R}Y$.

(U2) both $\bar{R}X$ and $\bar{R}Y$ are U or not-U together

Taking different combinations of the two types of conditions on lower approximations and the two types of conditions on upper approximations we get four kinds of approximate equalities of sets. This is summarized in the following table:

Upper approximation / Lower approximation	$\bar{R}X = \bar{R}Y$	$\bar{R}X$ and $\bar{R}Y$ are U or $\neq U$ together
$\underline{R}X = \underline{R}Y$	Rough Equalities	Approximate rough equalities
$\underline{R}X$ and $\underline{R}Y$ are ϕ or $\neq \phi$ together	Approximate rough equivalences	Rough Equivalences

Let T(i, j) ; i = 1, 2 and j = 1,2 denote the (i, j)th position in the above table. Then

(2.1.1) three different types approximate equalities called rough equalities were introduced by Novotny and Pawlak ([1, 2, 3]) taking T(1,1) into consideration.

(2.1.2) three different types of approximate equalities called rough equivalences were introduced by Tripathy, Mitra and Ojha taking T(2, 2) into consideration.

In this paper, we introduce two other types of approximate equalities as defined below.

(2.1.3) We shall say that two sets X and Y are approximately rough equivalent if they satisfy the conditions specified by the row and column heads for T(2, 1).

(2.1.4) We shall say that two sets X and Y are approximately rough equal if they satisfy the conditions specified by the row and column heads for T(1, 2).

In the next few sections, it is our aim to analyse these four kinds of approximate equalities.

2.2. Rough Equalities of Sets

As mentioned above, Novotny and Pawlak ([1,2,3]) introduced three types of approximate equalities, extending the idea of mathematical equality of two sets. We state these definitions below.

Definition 2.2.1: Let $K = (U, \Re)$ be a knowledge base, $X, Y \subseteq U$ and $R \in IND(K)$. We say that

(2.2.1) two sets X and Y are bottom R-equal $(X \eqsim_R Y)$ if and only if L1 holds true.

(2.2.2) two sets X and Y are top R-equal $X \simeq_R Y$ if and only if U1 holds true.

(2.2.3) two sets X and Y are R-equal $(X \approx_R Y)$ if $(X \eqsim_R Y)$ and $X \simeq_R Y$.

Equivalently, L1 and U1 both hold true.

For simplicity, we drop the suffix R in the above notations. It can be easily verified that the relations bottom R-equal, top R-equal and R-equal are equivalence relations over *P(U)*, the power set of *U*. The concept of approximate equality of sets refers to the topological structure of the compared sets but not the elements they consist of. Thus sets having significantly different elements may be rough equal. In fact, if $X \eqsim Y$ then $\underline{R}X = \underline{R}Y$ and as $X \subseteq \underline{R}X$, $Y \subseteq \underline{R}Y$ X and Y can differ in elements of $X - \underline{R}X$ and $Y - \underline{R}Y$. However, it is easy to check that two sets X and Y may be R-equal in spite of $X \cap Y = \phi$.

As noted by Pawlak ([5], p.26), rough equality of sets is of relative character, that is things are equal or not equal from our point of view depending on what we know about them. So, in a sense the definition of rough equality refers to our knowledge about the universe.

2.2.1. Basic Properties of Rough Equalities

The following properties of rough equalities are well known (see for instance [5]):

(2.2.1.1) $X \eqsim Y$ if and only if $X \cap Y \eqsim X$ and $X \cap Y \eqsim Y$.

(2.2.1.2) $X \simeq Y$ if and only if $X \cup Y \simeq X$ and $X \cup Y \simeq Y$.

(2.2.1.3) If $X \simeq X'$ and $Y \simeq Y'$ then $X \cup Y \simeq X' \cup Y'$.

(2.2.1.4) If $X \eqsim X'$ and $Y \eqsim Y'$ then $X \cap Y \eqsim X' \cap Y'$.

(2.2.1.5) If $X \simeq Y$ then $X \cup -Y \simeq U$.

(2.2.1.6) If $X \eqsim Y$ then $X \cap -Y \eqsim \phi$.

(2.2.1.7) If $X \subseteq Y$ and $Y \simeq \phi$ then $X \simeq \phi$.

(2.2.1.8) If $X \subseteq Y$ and $X \simeq U$ then $Y \simeq U$.

(2.2.1.9) $X \simeq Y$ if and only if $-X \eqsim -Y$.

(2.2.1.10) If $X \eqsim \phi$ or $Y \eqsim \phi$ then $X \cap Y \eqsim \phi$.

(2.2.1.11) If $X \simeq U$ or $Y \simeq U$ then $X \cup Y \simeq U$.

In the following two properties of lower and upper approximations of rough sets, we find that inclusions hold and equalities do not hold true in general:

(2.2.1.12) $\underline{R}X \cup \underline{R}Y \subseteq \underline{R}\left(X \cup Y\right)$ and

(2.2.1.13) $\bar{R}\left(X \cap Y\right) \subseteq \bar{R}X \cap \bar{R}Y$.

The following results ([10]) provide necessary and sufficient conditions for equalities to hold in (2.2.1.12) and (2.2.1.13). In these results we take $\{E_1, E_2, ..., E_n\}$ as a partition of a universe *U* with respect to an equivalence relation *R* and $X_1, X_2, ..., X_m$ are subsets of *U*.

Theorem 2.2.1.1: We have

(2.2.1.14) $\bigcup_{i=1}^{m} \underline{R}(X_i) \subset \underline{R}\left(\bigcup_{i=1}^{m} X_i\right)$ if and only if there exists at least one E_j such that $X_i \cap E_j \subset E_j$ for $i = 1, 2, ..., m$ and $\bigcup_{i=1}^{m}(X_i) \supseteq E_j$.

Corollary 2.2.1.1: Equality holds in (2.2.2.14) if and only if there exists no E_j such that $X_i \cap E_j \subset E_j$, $i = 1, 2, ..., m$ and $\bigcup_{i=1}^{m} X_i \supseteq E_j$.

Theorem 2.2.1.2: We have

(2.2.1.15) $$\bar{R}\left(\bigcap_{i=1}^{m} X_i\right) \subset \bigcap_{i=1}^{m} \bar{R}(X_i)$$

If and only if there exist at least one E_j such that $X_i \cap E_j \neq \phi$, $i = 1, 2, ..., m$ and $\left(\bigcap_{i=1}^{m} X_i\right) \cap E_j = \phi$.

Corollary 2.2.1.2: Equality holds in (2.2.2.15) if and only if there is no E_j such that $X_i \cap E_j \neq \phi$, $i = 1, 2, ..., m$ and $\left(\bigcap_{i=1}^{m} X_i\right) \cap E_j = \phi$.

It is noted in [5] that the properties (2.2.1.1) to (2.2.1.11) fail to hold if $\overset{\sim}{=}$ is replaced by $\underset{\sim}{=}$ or vice versa. However, it was shown in [10] that

I. The properties (2.2.1.7) to (2.2.1.11) hold true under the interchange.

II. The properties (2.2.1.5) and (2.2.1.6) holds true under interchange if $BN_R(Y) = \phi$.

III. (i) The properties (2.2.1.1) and (2.2.1.4) hold under interchange if conditions of Corollary 2.2.1.2 hold with m = 2.

(ii) The properties (2.2.1.2) and (2.2.1.3) hold if conditions of Corollary 2.2.1.1 hold with m = 2.

2.3. Rough Equivalences of Sets

Definition 2.3.1: We say that two sets *X* and *Y* are *bottom R-equivalent* if and only if both L2 holds and we write that *X* is b_eqv. to *Y*. We put the restriction here that for bottom R-equivalence of *X* and *Y* either both $\underline{R}X$ and $\underline{R}Y$ are equal to *U* or none of them is equal to *U*.

Definition 2.3.2: We say that two sets *X* and *Y* are *top R-equivalent* if and only if U2 holds true and we write that X is t_eqv. to *Y*. We put the restriction here that for top R-equivalence of *X* and *Y* either both $\bar{R}X$ and $\bar{R}Y$ are equal to ϕ or none of them is equal to ϕ.

Definition 2.3.3: We say that two sets *X* and *Y* are *R-equivalent* if and only if *X* and *Y* satisfy both L2 and U2 simultaneously and we write that X is eqv. to *Y*. We would like here to note that when two sets *X* and *Y* are R-equivalent, the restrictions in 2.2.1 and 2.2.2 become redundant.

2.3.1. Basic Properties of Rough Equivalence

In this section we state some of the properties of rough equivalences of sets obtained in [10]. These properties are similar to those of rough equalities. Some of these properties which do not hold in full force, sufficient conditions have been obtained. Also, the necessity of such conditions is verified. To state these properties, we need the concepts of different rough inclusions ([5]) and rough comparisons ([10]).

Definition 2.3.1.1: Let K = (U, $\mathfrak{R}$) be a knowledge base, X, Y $\subseteq$ U and R $\in$ IND(K). Then

(i) We say that X is *bottom R-included* in Y ($X \underset{\sim}{\subseteq}_R Y$) if and only if $\underline{R}X \subseteq \underline{R}Y$.

(ii) We say that X is *top R-included* in Y $\left(X \tilde{\subset}_R Y \right)$ if and only if $\bar{R}X \subseteq \bar{R}Y$.

(iii) We say that X is *R-included* in Y$\left(X \tilde{\underset{\sim}{\subseteq}}_R Y \right)$ if and only if $X \underset{\sim}{\subseteq}_R Y$ and $X \tilde{\subset}_R Y$

Definition 2.3.1.2: (i) we say X, Y $\subseteq$ U are *bottom rough comparable* if and only if $X \underset{\sim}{\subseteq} Y$ or $Y \underset{\sim}{\subseteq} X$ holds.

(ii) We say X, Y $\subseteq$ U are *top rough comparable* if and only if $X \tilde{\subset} Y$ or $Y \tilde{\subset} X$ holds.

(iii) We say X, Y $\subseteq$ U are *rough comparable* if and only if X and Y are both top and bottom rough comparable.

Property 2.3.1.1: (i) If $X \cap Y$ is b_eqv. to both *X* and *Y* then *X* is b_eqv. to *Y*.

(ii) The converse of (i) is not true in general and an additional condition that is sufficient but not necessary for the converse to be true is that X and Y are bottom rough comparable.

Property 2.3.1.2: (i) If $X \cup Y$ is t_eqv. to both *X* and Y then *X* is t_eqv. to *Y*.

(ii) The converse of (i) is not true in general and an additional condition that is sufficient but not necessary for the converse to be true is that X and Y are top rough comparable.

Property 2.3.1.3: (i) If *X* is t_eqv. to X' and *Y* is t_eqv. to Y' then it may or may not be true that $X \cup Y$ is t_eqv. to $X' \cup Y'$.

(ii) A sufficient but not necessary condition for the result in (i) to be true is that X and Y are top rough comparable and X' and Y' are top rough comparable.

Property 2.3.1.4: (i) X is b_eqv. to X' and Y is b_eqv. to Y' may or may not imply that $X \cap Y$ is b_eqv. to $X' \cap Y'$.

(ii) A sufficient but not necessary condition for the result in (i) to be true is that X and Y are bottom rough comparable and X' and Y' are bottom rough comparable.

Property 2.3.1.5: (i) X is t_eqv. to Y may or may not imply that $X \bigcup (-Y)$ is t_eqv. to U.
(ii) A sufficient but not necessary condition for result in (i) to hold is that $X \eqsim Y$.

Property 2.3.1.6: (i) X is b_eqv. to Y may not imply that $X \bigcap (-Y)$ is b_eqv. to ϕ.
(ii) A sufficient but not necessary condition for the result in (i) to hold true is that X□ Y.

Property 2.3.1.7: If $X \subseteq Y$ and Y is b_eqv. to ϕ then X is b_eqv. to ϕ.

Property 2.3.1.8: If $X \subseteq Y$ and X is t_eqv. to U then Y is t_eqv. to U.

Property 2.3.1.9: X is t_eqv. to Y if and only if –X is b_eqv. to –Y.

Property 2.3.1.10: X is b_eqv. to ϕ, Y is b_eqv. to ϕ $\Rightarrow X \bigcap Y$ is b_eqv. to ϕ.

Property 2.3.1.11: If X is t_eqv. to U or Y is t_eqv. to U then $X \bigcup Y$ is t_eqv. to U.

2.3.2. Replacement Properties for Rough Equivalence

In parallel to the properties of interchange for rough equalities, the following properties were proved in [10].

Property 2.3.2.1: (i) If $X \cap Y$ is t_eqv. to both X and Y then X is t_eqv. Y.
(ii) The converse of (i) is not true in general and an additional condition that is sufficient but not necessary for the converse to be true is that conditions of Corollary 2.2.2.2 hold with m = 2.

Property 2.3.2.2: (i) $X \cup Y$ is b_eqv. to X and $X \cup Y$ is b_eqv. to Y then X is b_eqv. to Y
(ii) The converse of (i) is not true in general and an additional condition that is sufficient but not necessary for the converse to be true is that the condition of corollary 2.2.2.1 holds with m = 2.

Property 2.3.2.3: (i) X is b_eqv. to X' and Y is b_eqv. to Y' may not imply $X \cup Y$ is b_eqv. to $X' \cup Y'$.
(ii) A sufficient but not necessary condition for the conclusion of (i) to hold is that the conditions of corollary 2.2.2.2 are satisfied for both X, Y and X', Y' separately with m = 2.

Property 2.3.2.4: (i) X is t_eqv. to X' and Y is t_eqv. to Y' may not necessarily imply that $X \cap Y$ is t_eqv. to $X' \cap Y'$.
(ii) A sufficient but not necessary condition for the conclusion in (i) to hold is that the conditions of corollary 2.2.2.1 are satisfied for both X, Y and X', Y' separately with m = 2.

Property 2.3.2.5: X is b_eqv. to Y may or may not imply $X \cup -Y$ is b_eqv. to U.

Property 2.3.2.6: X is t_eqv. to Y may or may not imply $X \cap -Y$ is t_eqv. to ϕ.

Property 2.3.2.7: If $X \subseteq Y$ and Y is t_eqv. to ϕ then X is t_eqv. to ϕ.

Property 2.3.2.8: If $X \subseteq Y$ and X is b_eqv.to U. then Y is b_eqv. to U.

Property 2.3.2.9: X is b_eqv. to Y if and only if $-X$ is t_eqv. to $-Y$.

Property 2.3.2.10: X is t_eqv. to ϕ and Y is t_eqv. to $\phi \Rightarrow X \cap Y$ is t_eqv. to ϕ.

Property 2.3.2.11: X is b_eqv. to U and Y is b_eqv. to $U \Rightarrow X \cup Y$ is b_eqv. to U.

2.4. Examples

In this section, we shall consider the two examples considered in [10] and [11] to illustrate the superiority of the concepts of rough equivalence over rough equalities.

2.4.1. Universe of Cattle

Let us consider the cattle in a locality as our universal set C. We define a relation R over C by x R y if and only if x and y are cattle of the same kind. This is an equivalence relation decomposes the universe into disjoint equivalence classes. Suppose for example, C = {Cow, Buffalo, Goat, Sheep, Bullock}. Let X and Y be the set of cattle owned by two persons P_1 and P_2 in the locality. We cannot talk about the equality of X and Y in the usual sense as the cattle cannot be owned by two different people. Similarly, we cannot talk about the rough equality of X and Y except the trivial case when both the persons do not own any cattle.

There different possibilities for approximate equality or otherwise of X and Y to hold have been discussed in [ ] and are categorized under six cases as follows:

Case (i). $\bar{R}X$, $\bar{R}Y$ are not U and $\underline{R}X$, $\underline{R}Y$ are ϕ. That is P_1 and P_2 both have some kind of cattle but do not have all cattle of any kind in the locality. So, they are equivalent.

Case (ii) $\bar{R}X$, $\bar{R}Y$ are not U and $\underline{R}X$, $\underline{R}Y$ are not ϕ. That is P_1 and P_2 both have some kind of cattle and have all cattle of some kind in the locality. So, they are equivalent.

Case (iii) $\bar{R}X$, $\bar{R}Y$ are U and $\underline{R}X$, $\underline{R}Y$ are ϕ. That is P_1 and P_2 both cattle of all kinds but do not have all cattle of any kind in the locality. So, they are equivalent.

Case (iv) $\bar{R}X$, $\bar{R}Y$ are U and $\underline{R}X$, $\underline{R}Y$ are not ϕ. That is P_1 and P_2 both have all kinds of cattle and also have all cattle of some kind in the locality. So, they are equivalent.

There are two different cases under which we can talk about the non- equivalence of P1 and P2.

Case (v) one of $\bar{R}X$ and $\bar{R}Y$ is U and the other one is not. Then, out of P1 and P_2 one has cattle of all kinds and the other one does not have so. So, they are not equivalent.

Case (vi) Out of $\underline{R}X$ and $\underline{R}Y$, one is ϕ and the other one is not. Then, out of P1 and P_2 one does not have all cattle of any particular kind, where as the other has all cattle of at least one kind. So, they are not equivalent.

2.4.2. Universe of Shares

Let us consider the example of a stock exchange dealing with shares of different companies. The shares of different companies are of different denomination and are related to branches of the companies at different locations.

We take S as the set of shares of different companies under the control of the stock exchange. We define three relations R_1, R_2 and R_3 on S as follows:

1. For $x, y \in S,$ we say xR_1y if and only if *x* and *y* are of the same denomination.

2. For $x, y \in S,$ we say xR_2y if and only if *x* and *y* are from branches with the same location.

3. For $x, y \in S,$ we say xR_3y if and only if *x* and *y* belong to the same company.

It is easy to check that each of R_1, R_2 and R_3 is an equivalence relation on *S*. Let P_1 and P_2 be two persons having shares in the stock exchange. S_1 and S_2 be the set of shares owned by them respectively. Ordinary set equality and rough equality notions cannot be applied to the sets S_1 and S_2 to compare the two persons from the ownership of shares point of view as the same share cannot be owned by more than one person in any stock exchange. However, from our common sense point of view and real life experience we know that two persons can be considered equivalent (or indistinguishable). For example, persons having shares from the same companies are equivalent to each other. Similarly, persons having shares of same denomination are equivalent. Also, persons having shares of branches of same location of different companies can be considered as equivalent.

Moving further, persons having shares of same company and same denominations can be considered as equivalent and persons having shares of same companies of same denominations and from branches of same location can also be considered as equivalent to each other. Mathematically, we are considering the intersections of R_1, R_2 and R_3 pair wise or all of them taken at a time, which are known to be equivalence relations as intersection of any number of equivalence relations is an equivalence relation.

Let S be a stock exchange having the set of shares $S = \{s_1, s_2, s_3, ..., s_{50}\}$. Suppose these shares are of three denominations of values Rs.10, Rs.15 and Rs.20 (we mention as V_1, V_2 and V_3). Then we define:

$$R_1 = \{V_1 = \{s_1, s_4, s_7, s_{10}, s_{13}, s_{16}, s_{19}, s_{22}, s_{25}, s_{28}, s_{31}, s_{34}, s_{37}, s_{40}, s_{43}, s_{46}, s_{49}\},$$
$$V_2 = \{s_2, s_5, s_8, s_{11}, s_{14}, s_{17}, s_{20}, s_{23}, s_{26}, s_{29}, s_{32}, s_{35}, s_{38}, s_{41}, s_{44}, s_{47}, s_{50}\},$$
$$V_3 = \{s_3, s_6, s_9, s_{12}, s_{15}, s_{18}, s_{21}, s_{24}, s_{27}, s_{30}, s_{33}, s_{36}, s_{39}, s_{42}, s_{45}, s_{48}\}\} .$$

The shares belong to branches of different companies at four locations L_1, L_2, L_3 and L_4 . So, we define: $R_2 = \{L_1 = \{s_1, s_5, s_9, s_{13}, s_{17}, s_{21}, s_{25}, s_{29}, s_{33}, s_{37}, s_{41}, s_{45}, s_{49}\},$

$$L_2 = \{s_2, s_6, s_{10}, s_{14}, s_{18}, s_{22}, s_{26}, s_{30}, s_{34}, s_{38}, s_{42}, s_{46}, s_{50}\},$$
$$L_3 = \{s_3, s_7, s_{11}, s_{15}, s_{19}, s_{23}, s_{27}, s_{31}, s_{35}, s_{39}, s_{43}, s_{47}\},$$
$$L_4 = \{s_4, s_8, s_{12}, s_{16}, s_{20}, s_{24}, s_{28}, s_{32}, s_{36}, s_{40}, s_{44}, s_{48}\}\} .$$

The exchange contains shares of five different companies C_1, C_2, C_3, C_4 and C_5 . We define

$$R_3 = \{C_1 = \{s_1, s_6, s_{11}, s_{16}, s_{21}, s_{26}, s_{31}, s_{36}, s_{41}, s_{46}\},$$
$$C_2 = \{s_2, s_7, s_{12}, s_{17}, s_{22}, s_{27}, s_{32}, s_{37}, s_{42}, s_{47}\},$$

$$C_3 = \{s_3, s_8, s_{13}, s_{18}, s_{23}, s_{28}, s_{33}, s_{38}, s_{43}, s_{48}\},\ C_4 = \{s_4, s_9, s_{14}, s_{19}, s_{24}, s_{29}, s_{34}, s_{39}, s_{44}, s_{49}\},$$
$$C_5 = \{s_5, s_{10}, s_{15}, s_{20}, s_{25}, s_{30}, s_{35}, s_{40}, s_{45}, s_{50}\}\} .$$

Let P_1 , P_2 , S_1 and S_2 be as above.

Case 1: Let $S_1 = \{s_1, s_5, s_{10}\}$ and $S_2 = \{s_2, s_4, s_{16}, s_{23}\}$ .Then from the point of view of holding shares of similar denominations, both P_1 and P_2 do not hold all shares of any particular denomination and they do not hold shares of all denominations. So they are rough equivalent.

Case 2: Let $S_1 = \{s_1, s_5, s_6, s_9, s_{10}\}$ and $S_2 = \{s_4, s_8, s_{11}, s_{13}\}$ .Then P_1 and P_2 are bottom equivalent as none of them has all the shares of any particular denomination. But they are not totally rough equivalent as P_1 has shares of denomination Rs.20 where as P_2 does not hold any such share.

Case 3: Let $S_1 = \{s_1, s_4, s_7, s_{10}, s_{13}, s_{16}, s_{19}, s_{22}, s_{25}, s_{28}, s_{31}, s_{34}, s_{37}, s_{40}, s_{43}, s_{46}, s_{49}\}$ and $S_2 = \{s_3, s_6, s_9, s_{12}, s_{15}, s_{18}, s_{21}, s_{24}, s_{27}, s_{30}, s_{33}, s_{36}, s_{39}, s_{42}, s_{45}, s_{48}\}$.
Then from the point of view of holding shares of similar denominations, both P_1 and P_2 holds all the shares of a particular denomination and they do not hold shares of all denominations. So they are rough equivalent.

Case 4: Let $S_1 = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ and $S_2 = \{s_7, s_8, s_9, s_{10}, s_{11}, s_{12}\}$.
Then from the point of view of holding shares of similar denominations, both P_1 and P_2 do not holds all the shares of a particular denomination but they do hold at least one share of all denominations. So they are rough equivalent.

Case 5: Let
$S_1 = \{s_2, s_5, s_8, s_{11}, s_{14}, s_{17}, s_{20}, s_{23}, s_{26}, s_{29}, s_{32}, s_{35}, s_{38}, s_{41}, s_{44}, s_{47}, s_{48}, s_{49}, s_{50}\}$ and
$S_2 = \{s_1, s_4, s_7, s_{10}, s_{13}, s_{16}, s_{19}, s_{22}, s_{25}, s_{28}, s_{31}, s_{34}, s_{37}, s_{40}, s_{43}, s_{46}\}$.
Then P_1 holds all the shares of the denomination Rs.10 whereas P_2 does not have this property and P_1 holds some share of each denominations whereas P_2 has shares of denomination R.10 only. So P_1 and P_2 are not rough equivalent.

Similarly, we can show different rough equivalence for holding shares of different branches and different companies; that is with respect to the relations R_2 and R_3 respectively.
Let us consider two relations at a time, say R_1 and R_2 ; that is shares of same denomination and of same branches. By direct calculation, we have

$$R_1 \cap R_2 = \{\{s_1, s_{13}, s_{25}, s_{37}, s_{49}\}, \{s_{10}, s_{22}, s_{34}, s_{46}\}, \{s_7, s_{19}, s_{31}, s_{43}\}, \{s_{16}, s_{28}, s_{40}\}, \{s_5, s_{17}, s_{29}, s_{41}\},$$
$$\{s_2, s_{14}, s_{26}, s_{28}, s_{50}\}, \{s_{11}, s_{23}, s_{35}, s_{47}\}, \{s_8, s_{20}, s_{32}, s_{44}\}, \{s_9, s_{21}, s_{33}, s_{45}\},$$
$$\{s_6, s_{18}, s_{30}, s_{42}\}, \{s_3, s_{15}, s_{27}, s_{39}\}, \{s_{12}, s_{24}, s_{36}, s_{48}\}\}.$$

Case 6: Let $S_1 = \{s_1, s_{11}, s_{21}, s_{31}, s_{41}\}$ and $S_2 = \{s_2, s_{12}, s_{22}, s_{32}, s_{42}\}$.
Then P_1 and P_2 do not hold all shares of any particular branch and same denomination. Also, they do not have such share across all the branches and of same denomination. So they are rough equivalent.

Case 7: Let $S_1 = \{s_1, s_{13}, s_{25}, s_{37}, s_{49}, s_{50}\}$ and $S_2 = \{s_{11}, s_{12}, s_{24}, s_{36}, s_{48}\}$.

Then P_1 and P_2 hold all shares of same particular branch which are of same denomination. But, they do not have such share across all the branches and of same denomination. So they are rough equivalent.

Case 8: Let $S_1 = \{s_1, s_2, s_3, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{16}\}$
and $S_2 = \{s_{13}, s_{14}, s_{15}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{28}, s_{36}, s_{48}\}$.

Then from the point of view of same denomination and same branch office shares of the stock exchange P_1 and P_2 does not hold all shares of any particular branch and same denomination. But, they have such share across all the branches and of same denomination. So they are rough equivalent.

Case 9: Let $S_1 = \{s_{18}, s_{30}, s_{39}, s_{40}, s_{41}, s_{42}, s_{43}, s_{44}, s_{45}, s_{46}, s_{47}, s_{48}, s_{49}, s_{50}\}$
and $S_2 = \{s_{13}, s_{14}, s_{15}, s_{17}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{28}, s_{36}\}$.

Then P_1 and P_2 are bottom equivalent as none of them has all the shares of any particular branch and same denomination. But they are not totally rough equivalent as P_1 has at least one share across all the branches and of same denomination where as P_2 does not hold at least one share across all the branches and of same denomination.

Case 10: Let $S_1 = \{s_6, s_{18}, s_{30}, s_{42}, s_{39}, s_{40}, s_{41}, s_{43}, s_{44}, s_{45}, s_{46}, s_{47}, s_{48}, s_{49}, s_{50}\}$
and $S_2 = \{s_{13}, s_{14}, s_{15}, s_{17}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{28}, s_{36}\}$.

Then P_1 has all the shares of a particular branch of same denominations; whereas P_2 does not have this property. So, they are not bottom rough equivalent as far as holding shares of same branch and same denomination is concerned. Similarly, P_1 has shares of every branches of every denomination; whereas P_2 does not have this property. So, they are not top rough equivalent also.

Finally, we consider all three relations at a time. Then $R_1 \cap R_2 \cap R_3 = \{\{s_1\}, \{s_2\}, \{s_3\}, ..., \{s_{49}\}, \{s_{50}\}\}$.

We see that any two share holders are rough equivalent with respect to holding of shares of same denomination, same branch and same company is concerned. As, a share holder must have at least one share and when more than one share holders are there, none can have all shares, we see that for P_1 and P_2 we have only one case.

3. Approximate Rough Equivalence and Approximate Rough Equality of Sets

In this section we introduce the two other types of approximate equalities; which we call as approximately rough equivalent and approximately rough equal, out of which approximately rough equivalent is more general than the concept of rough equality and restricted than the concept of rough equivalence and also seems to be more novel than the two types of approximate equalities.

Definition 3.1: We say that two sets X and Y are approximately rough equivalent if and only if L2 and U1 hold true, that is when both $\underline{R}X$ and $\underline{R}Y$ are ϕ or not ϕ together and $\bar{R}X = \bar{R}Y$.

Definition 3.2: We say that two sets X and Y are approximately rough equal if and only if L1 and U2 hold true, that is when $\underline{R}X = \underline{R}Y$ and both $\overline{R}X$ and $\overline{R}Y$ are U or not U together.

4. Comparisons

In this section we shall provide a comparative study of the four different types of approximate equalities.

4.1. The condition L1, on interpretation states that two sets are lower approximately equal if and only if the two sets have the same lower approximation; that is both the sets must include exactly the same equivalence classes. This sounds alright in numerical examples. But, when we move to practicalities, it is found that the condition requires some elements of the universe to belong to both the sets; which seems to be unnatural. For example, taking the example of cattle into consideration, we cannot have cases where the same set of cattle belonging to two different persons. In the example of shares, any share belonging to two different people is not feasible. The examples show that in only rare and restricted cases, this property may be satisfied. Since we are using this property in case of both rough equal and approximately rough equal definitions, these two cases of rough equalities seem to have lesser utility than the other two.

4.2. No doubt the condition U2 provides freedom to define equality in a very approximate sense and is quite general than U1. But, sometimes it seems to be unconvincing. For example, let us take the cattle example into consideration. When P1 and P2 both are not equal to U, we say that they are top rough equivalent. Suppose a person has a cow and a buffalo in the locality and another has most of the other animals in the locality. Then they are top rough equivalent, by using this definition, which does not seem to be convincing. On the other hand, if we take U1 into consideration then the above two persons are not top rough equal. However, P1 and P2 shall be top rough equal if and only if they contain cattle of same kind, irrespective of their number and it satisfies common sense reasoning although it is approximate.

4.3. Let us now analyse the concept of approximate rough equivalence. It is neither unconvincing nor unnatural. Two persons can be said to be equal in this sense only when they have nonempty intersections with same equivalence classes and either include all or none of the elements of the equivalence classes separately. Since it is impossible for two persons to have the same set of cattle of any kind this is the best possible type of approximate equality from all angles.

4.4. The fourth type of approximate equality defined above has both the problems mentioned in 4.1 and 4.2. So this happens to be the worst type of approximate equality which we can talk about.

5. General Properties

In this section we shall consider the general properties of the two new types of approximate equalities introduced by us in this paper.

5.1. Approximate Rough Equivalence Properties

Since the definition of approximate rough equivalence is a combination of L2 and U1, the general properties are just to be picked up from those of the other types of approximate equalities. We mention them below.

The properties (2.2.1.2), (2.2.1.3), (2.2.1.5), (2.2.1.7), (2.2.1.8), (2.2.1.11), (2.3.1.1), (2.3.1.4), (2.3.1.6), (2.3.1.7) and (2.3.1.10) hold true.

5.2. Approximate Rough Equality Properties

Since the definition of approximate rough equivalence is a combination of L1 and U2, the general properties are just to be picked up from those of the other types of approximate equalities. We mention them below.

The properties (2.2.1.1), (2.2.1.4), (2.2.1.6), (2.2.1.10), (2.3.1.2), (2.3.1.3), (2.3.1.5), (2.3.1.8), (2.3.1.9), and (2.3.1.11) hold true.

It may be noted that the property (2.2.1.9) does not hold for any of the above two types of approximate equalities.

6. Properties with Replacements

In this section we shall consider the properties with lower approximate rough equivalence replaced with upper approximate rough equivalence and vice versa for the two new types of approximate equalities introduced by us in this paper.

6.1. Approximate Rough Equivalence Properties

Since the definition of approximate rough equivalence is a combination of L2 and U1, the general properties are just to be picked up from those of the other types of approximate equalities. We mention them below.

(i) The properties (2.2.1.7), (2.2.1.8), (2.2.1.11) hold under the interchange. Also, the properties (2.3.2.2), (2.3.2.3), (2.3.2.5), (2.3.2.8) and (2.3.2.11) hold true.

(ii) The property (2.2.1.5) holds true under interchange if $BN_R(Y) = \phi$.

(iii) The properties (2.2.1.2) and (2.2.1.3) hold if conditions of Corollary 2.2.2.1 hold with m = 2.

6.2. Approximate Rough Equality Properties

Since the definition of approximate rough equivalence is a combination of L1 and U2, the general properties are just to be picked up from those of the other types of approximate equalities. We mention them below.

(i) The property (2.2.1.10) hold true under the interchange. Also, the properties (2.3.2.4), (2.3.2.6), (2.3.2.7) and (2.3.2.10) hold true.

(ii) The property (2.2.1.6) holds true under interchange if $BN_R(Y) = \phi$.

(iii) The properties (2.2.1.1) and (2.2.1.4) hold under interchange if conditions of Corollary 2.2.2.2 hold with m = 2.

7. Conclusions

In this paper we introduced two more types of approximate equalities in addition to the two earlier types of approximate equalities introduced and studied by Novotny and Pawlak [1, 2, 3] and Tripathy et al.[10]. We also provided a comparative study of these four types of approximate equalities. In fact, we established that the two earlier types of approximate equalities have some practical problems in their application to real life situations. As far as the two new types of approximate equalities are concerned approximate rough equivalence is better than approximate rough equality. Overall among the four types of approximate equalities it is established that rough equivalence is the best from the application point of view and is the one having enough novelty in it.

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Author

Dr.B.K.Tripathy is a senior professor in the school SCSE of VIT University, Vellore, India since 2007. He has produced 10 PhDs so far and has published over 115 technical papers in various international/National Journals and International/ National conference proceedings/Springer Book Chapters. He is in the editorial board of several journals and is a reviewer of many other journals including some of Elsevier and Springer publications. He is a life member of several international bodies including IEEE, IST, IRSS, AISTC, WSEAS, CSI and IMS. His current research interest includes Fuzzy Set Theory and Systems, Rough Sets and Knowledge Engineering, Granular Computing, Soft Computing, Social Network Analysis, Data Clustering Techniques and Concept Analysis.