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A Study of 1D Quadratic Map

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Abstract

We study in this paper, a non linear model of one dimensional discrete dynamical system, we propose a model of quadratic map and we analyze its bifurcation diagram, the iterative equation of the map is deduced from the relation of the laminar velocity of a fluid flow in circular section where the velocity depends on the position of the particles of the fluid. The growth parameter of the iterative equation corresponds to the maximal velocity of the flow. Numerical results show that the upper bound of the growth parameter is sqrt(2) where the bifurcation diagram and Lyapunov exponent are presented. In the second part, we analyze some properties of generating binary sequence in the chaotic phase by numerically searching for threshold that maximizes the information entropy, the obtained result for the optimal threshold shows that the binary sequence is not correlated.

Keywords*: Quadratic map, one dimensional, laminar velocity, bifurcation, binary sequence, correlation.*

1. Introduction

In discrete dynamical systems [1], the logistic map [2] is well known non linear model and second order iterative equation that puts into evidence several properties such as period doubling and chaotic behavior of the sequence defined by the relation $x_{n+1} = \mu x_n (1 - x_n)$ with growth parameter $0 \le \mu \le 4$, its applications span several fields including encryption systems [3,4] and chaotic wireless communications [5]. The analysis of the properties of logistic map is based on bifurcation diagram which is a map showing stable and chaotic behavior of the sequence with respect to the growth parameter μ. Several extensions of the logistic map were studied by changing its properties or by introducing new parameters. In research presented in [6], the authors analyzed the effect of the randomness of the growth parameter μ which was modeled by bounded random variable, where the authors found a new chaotic behavior that depends on the introduced bounds of μ.

In research [7], the logistic map was modified by adding a parameter to the quadratic term of the sequence $x_{n+1} = f(x_n, \mu)$ where the analysis of the obtained results of the sequence x_n can be applied in random-bit generation. In [8], a new logistic map based on the relation of the information entropy was studied where the bifurcation diagram resembles that of the standard logistic map with growth parameter in the range $[0, e]$, it was shown that the bifurcation diagram corresponds to the diagram of optimum velocity-density in normalized Greenberg's model of traffic flow.

The logistic map was also applied to study the Greenshields' model of traffic flow [9] where the velocity varies linearly with the increasing density which the

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number of vehicles per length of the considered section of the lane. The authors in [10], applied the logistic map as discrete model of traffic flow where the sequence x_n represents the occupancy and the growth parameter μ corresponds to the ratio of the free flow speed to average speed. This discrete model of logistic map permitted to study different conditions of traffic flow. Following the same context, we numericaly study a model of one dimensional quadratic map. Based on the equation of the velocity profile of fluid in circular geomerty [11] where the velocity depends on the position of the particles in the cross section [12], the iterative equation of the map given by the relation $x_{n+1} = \mu(1-x_n^2)$ is inspired from the relation of the velocity profile $v(v_{\max}, r)$ where the maximal velocity $v_{\max}$ corresponds to the growth parameter μ. We present the bifurcation diagram and its corresponding Lyapunov exponent.

2. Quadratic Map

The most well known model of quadratic maps is the logistic map [2] which is model of non linear dynamical system, the bifurcation diagram puts into evidence several properties including period doubling and chaotic behavior of the sequence, its corresponding iterative relation is given by $x_{n+1} = \mu x_n(1-x_n)$ defined by the growth rate $\mu \in [0,4]$. In the interval $[3.5, 4]$, the sequence x_n is characterized by chaotic behavior which can be measured using Lyapunov exponent [13]. Several other quadratic maps were proposed for the purpose of studying non linear phenomena, for example in [14], the authors proposed two models of quadratic map defined by he following relations:

$$\begin{cases} x_{n+1} = 1 - \mu x_n^2 \\ x_{n+1} = x_n^2 + \mu \end{cases} \tag{1}$$

Where the bifurcation diagrams are also characterized by chaotic behavior for defined intervals of the parameter μ. Based on the models of the above equation, we propose a standard transformation of the quadratic map where the expression of the sequence is based on the velocity profile of fluid flow in circular geometry. In fluid dynamics, the laminar flow of a fluid in pipe with radius R is described by a velocity that is maximal at the center [12] and equals zero in the borders of the pipe, the relation of the velocity profile and the radius is given by the following equation:

$$v(r) = v_{\max}\left(1-\left(\frac{r}{R}\right)^2\right) \tag{2}$$

where $v_{\max}$ is the maximal velocity of the flow. Based on this relation of the laminar velocity, we study the one dimensional and non linear iterative process defined by $r_{n+1} = v(r_n, v_{\max})$, which is defined by initial condition r_0 and characterized by single parameter $v_{\max}$, the expression of the sequence is given by:

$$r_{n+1} = v(r_n, v_{\max}) = v_{\max}\left(1-\left(\frac{r_n}{R}\right)^2\right) \tag{3}$$

For simplicity, we change the notations of the variables and normalize the range of the displacement r_n by setting $\mu = v_{\max}$ and $R = 1$, thus we obtain in the usual notation a model of quadratic map defined by:

$$x_{n+1} = \mu(1 - x_n^2) \qquad (4)$$

With initial condition $x_1 \in [0,1]$. We numerically study the behavior of the sequence x_n with respect to the parameter μ. We consider the sampling rate of $d\mu = 0.004$, we randomly choose an initial value x_1 in the range $[0,1]$, the sequence is computed using $N = 10000$ iterations where we take the last 5000 samples to ensure that the system reaches a stationary states. In figure.1, we present the obtained bifurcation diagram.

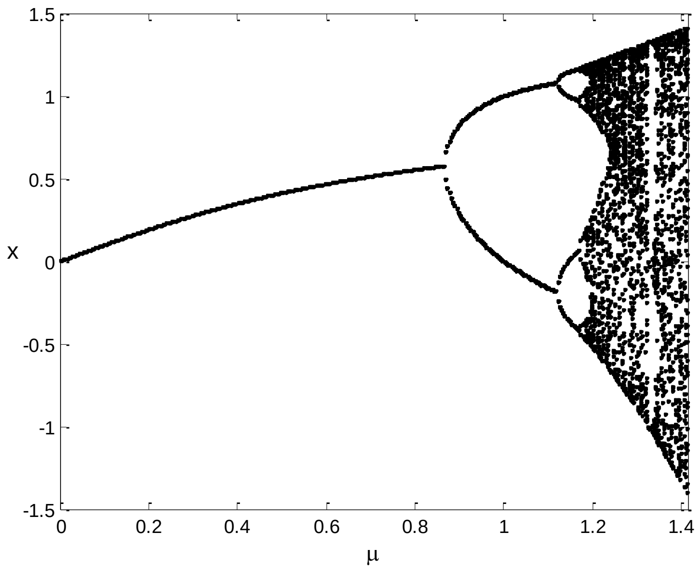

Figure 1. Bifurcation Diagram of Quadratic Map $x_{n+1} = \mu(1 - x_n^2)$

By changing the interval of the growth parameter, we remark that it is defined by the range $\left[0, \sqrt{2}\right]$ and the range of the sequence is $\left[-\sqrt{2}, \sqrt{2}\right]$. The first bifurcation point is approximately located in $(0.8, 0.5)$ and the chaotic region is approximately in the range $\left[1.2, \sqrt{2}\right]$. Next, we evaluate the corresponding Lyapunov exponent [13]. For N points of the sequence x_n, Lyapunov exponent is defined by the following relation:

$$\lambda(\mu) = \frac{1}{N}\sum_{n=1}^{N} \log\left|\left(\frac{\partial f(x)}{\partial x}\right)_{x_n}\right| \qquad (5)$$

Where $f(x) = \mu(1 - x^2)$, the derivative is $\partial f(x)/\partial x = -2\mu x$. The corresponding Lyapunov exponent is given by:

$$\lambda(\mu) = \frac{1}{N}\sum_{n=1}^{N} \log(2\mu|x_n|) \qquad (6)$$

In figure.2, we present the Lyapunov exponent where positive values correspond to the chaotic behavior of the quadratic map.

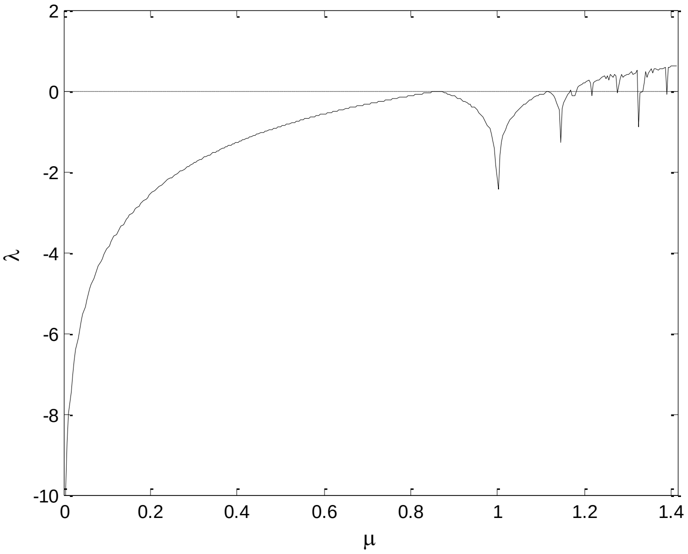

Figure 2. Lyapunov Exponent of Quadratic Map $x_{n+1} = \mu(1 - x_n^2)$

Next, we present in figure.3, the time series using the same initial condition $x_1 = 0.15$ with several values of the growth parameter, we remark the difference between the patterns of the sequences x_n in terms of periodicity and randomness, which can be analyzed using power spectra.

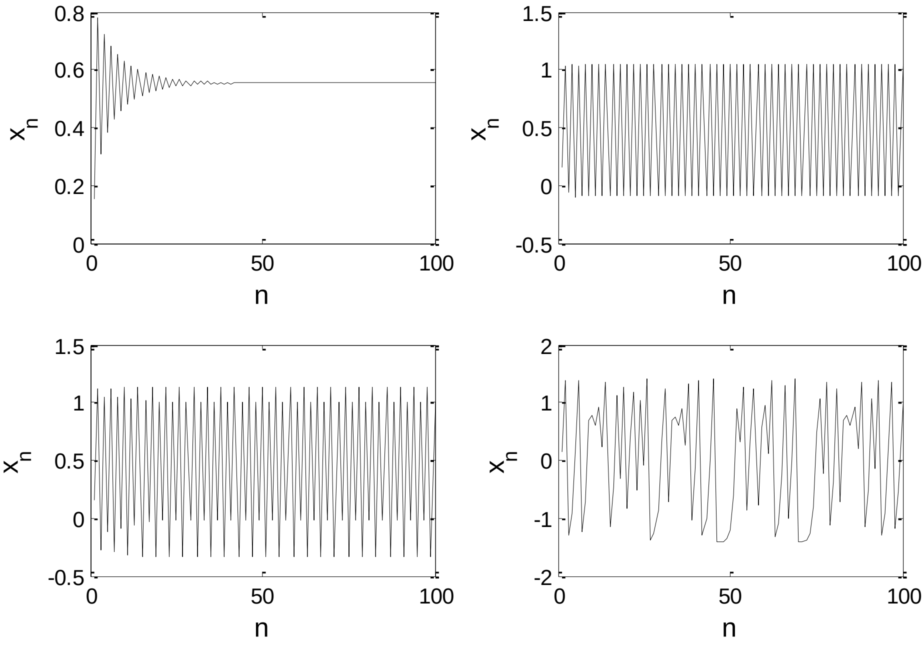

Figure 3. Time Series of Quadratic Map $x_{n+1} = \mu(1 - x_n^2)$ **with** $x_1 = 0.15$ **and** **(** $\mu = 0.8$**,** $\mu = 1.05$**,** $\mu = 1.14$**,** $\mu = \sqrt{2}$ **)**

As comparative study, we present in table.1, some characteristic of well known maps in the literature including the logistic map [2], two models of quadratic map [14], ecology [15], logarithmic [8] and lorentzian [16] maps where theirs bifurcation points are approximated numerically.

Table 1. Some Characteristics of Different Maps

Map function	Expression	Bifurcation point
Ecology map	$x_{n+1} = x_n e^{\mu(1-x_n)}$	(2.0,1.0)
Logistic map	$x_{n+1} = \mu x_n (1 - x_n)$	(3,0.67)
Quadratic map 1	$x_{n+1} = 1 - \mu x_n^2$	(0.75,0.68)
Quadratic map 2	$x_{n+1} = x_n^2 + \mu$	(-0.7,-0.5)
Lorentzian map	$x_{n+1} = (1 + \mu x_n^2)^{-1}$	(4,0.5)
Logarithmic map	$x_{n+1} = -\mu x_n \log(x_n)$	(2,0.6)
Proposed map	$x_{n+1} = \mu(1 - x_n^2)$	(0.8,0.5)

The quadratic maps presented in the above table are standard transformations of the logistic map, such that they can also be used in several applications where the logistic map is applied. Among the possible applications of the logistic map is random number generation [15], for $\mu = 4$, the sequence $x_{n+1} = 4x_n(1 - x_n)$ can be used to generate random numbers y_n uniformly distributed in the range $[0,1]$ using the following transformation:

$$y_n = \left(\frac{1}{\pi}\right)\cos^{-1}(1 - 2x_n) \quad (7)$$

Where the initial value x_1 must be different than $1/4$, $1/2$, $3/4$ and 1, for these values the sequence produces periodic series [17]. Another application consists of generating binary sequences, as a second part of this study, we analyze some properties of generating a binary sequence of the quadratic map in the chaotic phase. This process requires a threshold for splitting the values of the sequence. We fix the growth parameter at $\mu = \sqrt{2}$, and we search for an optimal partition of x_n. the optimal threshold consists of dividing the values of the sequences into binary values $\left(-\sqrt{2}, \sqrt{2}\right)$ with the same probability, and this condition implies that entropy of x_n must be maximal, therefore we vary the threshold in the range $d \in \left[-\sqrt{2}, \sqrt{2}\right]$ with sampling rate $\delta = 0.01$ and we compute the corresponding entropy function. The variation of the information entropy as a function of threshold is presented in figure.4. We remark that the optimal value of the threshold is $d \approx 0$ where the entropy equals $\log(2)$. For this value of the threshold, the first and second order statistics of the binary sequence are $< y >= 0$ and $< y^2 >= 2$. The generated sequence is characterized by spectral and correlation properties, we analyze the corresponding correlation function which is defined by the relation:

$$f(\tau) = \frac{1}{N}\sum_{n=1}^{N-\tau} x_n x_{n+\tau} \quad (8)$$

For $\tau = 0,...,N-1$. In figure.5, we present the correlation function $f(\tau)$ which shows that the sequence is not correlated.

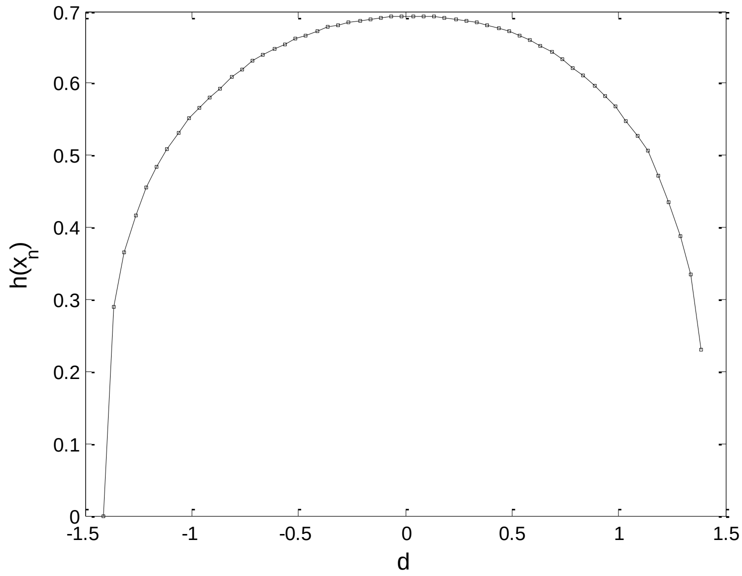

Figure 4. Shannon Entropy of Binary Sequence $h(x_n)$ with Respect to the Threshold d

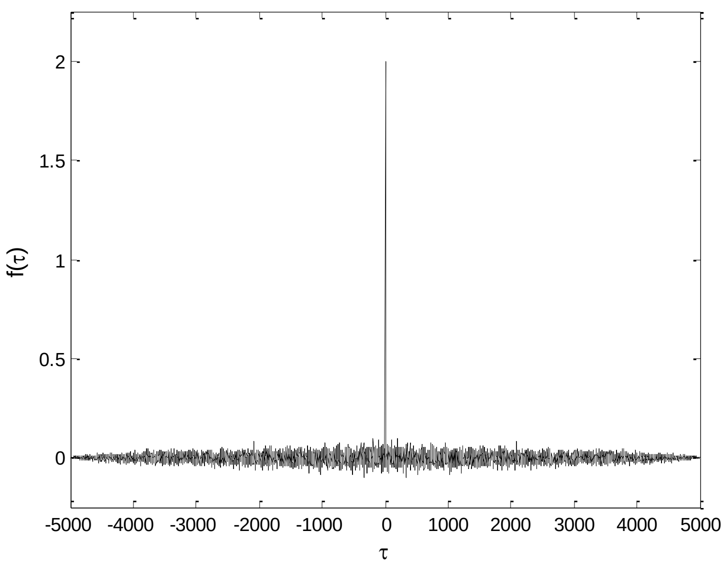

Figure 5. Autocorrelation Function $f(\tau)$ of the Binary Sequence x_n Obtained from Quadratic Map

The correlation function can be approximated by $f(\tau) \approx < y^2 > \delta(\tau) = 2\delta(\tau)$. Comparatively , for the same procedure of generating binary sequence with values $(0,1)$ using the logistic map with growth parameter $\mu = 4$ and optimal threshold $d = 0.5$, the computed function $f(\tau)$ shows that the sequence is correlated, therefore the proposed scheme of generating binary sequence using quadratic map $x_{n+1} = \mu(1 - x_n^2)$ is considered as more efficient for decreasing the correlation property.

4. Conclusion

In this paper, we have studied a one dimensional and non linear model of discrete dynamical system, we have proposed a type of quadratic map where the iterative equation of the sequence is based on the relation of the laminar velocity of a fluid flow in channel of circular geometry. The growth parameter of the iterative equation is equivalent to the maximal velocity of the flow. We have analyzed the bifurcation diagram where we numerically found the upper bound of the growth parameter and the region of the chaotic behavior using Lyapunov exponent. In the second part, the procedure of generating binary sequence with optimal partition using information entropy is presented. For the optimal threshold, the numerical results showed that the generated binary sequence is characterized by the absence of correlation.

Appendix

In this part, we present some programs for computing the bifurcation diagram, the corresponding Lyapunov exponent and the binary sequence using C programming language, the outputs of the programs are written in text files containing two columns, the first column is for the growth parameter μ and the second column corresponds to the sequence x_n and Lyapunov exponents $\lambda(\mu)$. For the binary sequence, the output consists of single column.

1. Program 1Dquadraticmap.c

```
#include<stdio.h>
#include<stdlib.h>
#include<math.h>

FILE *f;
int main()
{
   printf("Quadratic map x[n+1]=mu(1-x[n]^2)\n");
   f=fopen("data_sequence.txt","w");
   int i,N=10000;
   double mu;
   double x[N-1];
   for (mu=0;mu<=sqrt(2);mu+=0.004)
   {
      for(i=0;i<=N-1;i++)
      {
         x[i]=0;
      }
      x[0]=(double)rand()/RAND_MAX;
      for (i=0;i<=N-2;i++)
      {
         x[i+1]=mu*(1-x[i]*x[i]);
      }
```

```
            for (i=N/2-1;i<=N-1;i++)
            {
                fprintf(f,"%.4f\t%.4f\n",mu,x[i]);
            }
            fprintf(f,"\n");
        }
        fclose(f);
        return 0;
}
```

2. Program Lyapunov_quadratic.c

```
    #include<stdio.h>
    #include<stdlib.h>
    #include<math.h>

    FILE *f;
    int main()
    {
        printf("Lyapunov exponent of quadratic map x[n+1]=mu(1-x[n]^2)\n");
        f=fopen("data.txt","w");
        int i,N=10000;
        double mu;
        double x[N-1];
        for (mu=0.001;mu<=sqrt(2);mu+=0.004)
        {
            for(i=0;i<=N-1;i++)
            {
                x[i]=0;
            }
            x[0]=(double)rand()/RAND_MAX;;
            for (i=0;i<=N-2;i++)
            {
                x[i+1]=mu*(1-x[i]*x[i]);
            }
            double s=0,L=0;
            for (i=N/2-1;i<=N-1;i++)
            {
                if (x[i]<0)
                {
                    x[i]=-x[i];
                }

                s+=(double)log(2*mu*x[i]);
            }
            L=(s*2/N);
            fprintf(f,"%.4f\t%.4f\n",mu,L);

        }
        fclose(f);
        return 0;
}
```

3.Program binary_quadraticmap.c

```
#include<stdio.h>
#include<stdlib.h>
#include<math.h>

FILE *f;
int main ()
{
        printf("binary sequence generated by quadratic map \n");
        f=fopen("binary_sequence.txt","w");
        int i,N=10000;
        double mu=sqrt(2);
        double y[N-1],x[N/2-1];
        for (i=0;i<=N/2-1;i++)
        {
                x[i]=0;
        }
    for (i=0;i<=N-1;i++)
    {
        y[i]=0;
        }
    y[0]=(double)rand()/RAND_MAX;;
    for (i=0;i<=N-2;i++)
    {
        y[i+1]=mu*(1-y[i]*y[i]);
        }
    int j=0;
    for (i=N/2;i<=N-1;i++)
    {
        x[j]=y[i];
        j++;
        }
    for (i=0;i<=N/2-1;i++)
    {
        if (x[i]<0)
        {
                x[i]=-sqrt(2);
        }
            else
        {
                    x[i]=sqrt(2);
                }


                fprintf(f,"%.4f\n",x[i]);
        }
    fclose(f);
    return 0;
}
```


References

[1] S. Lynch, “Dynamical Systems with Applications using MATLAB”, Birkhäuser Basel, **(2014)**.
[2] M. Robert, "Simple mathematical models with very complicated dynamics", Nature, vol. 261, no. 5560, **(1976)**, pp. 459-467.
[3] A. Akhshani, A. Akhavan, S.-C. Lim and Z. Hassan, “An image encryption scheme based on quantum logistic map”, Communications in Nonlinear Science and Numerical Simulation, vol. 17, **(2012)**, pp. 4653-4661.
[4] L. Li-Hong, B. Feng-Ming and H. Xue-Hui, “New image encryption algorithm based on logistic map and hyper-chaos”, 2013 International Conference on Computational and Information Sciences, **(2013)**, pp. 713-716.
[5] Z. Xueyi, J. Lu, W. Kejun and L. Dianpu, “Logistic-map chaotic spread spectrum sequences under linear transformation”, Proceedings of the 3rd World Congress on Intelligent Control and Automation (Cat. No.00EX393), vol. 4, **(2000)**, pp.2464-2467.
[6] A. Khaleque and P. Sen, “Effect of randomness in logistic maps”, International Journal of Modern Physics C 26, **(2015)**, 1550086.
[7] J. Noymanee and W. San-Um, “A modified simple logistic chaotic map through exponential controller in nonlinear term”, 2015 Science and Information Conference (SAI), **(2015)**, pp. 533-537.
[8] Y. Khmou, “A numerical study of new logistic map”, Fluctuation and Noise Letters, **(2018)**.
[9] Y. M. Zhang and S. R. Qu, "Research on Chaotic Characteristics for Freeway Traffic Flow," 2009 International Conference on Measuring Technology and Mechatronics Automation, Zhangjiajie, Hunan, **(2009)**, pp. 559-562.
[10] Z. Yu-mei and Q. Shi-ru, "Chaotic property research for freeway traffic flow", 2009 Chinese Control and Decision Conference, Guilin, **(2009)**, pp. 1840-1843.
[11] B. R. Munson, D. F. Young and T. H. Okiishi, Fundamentals of Fluid Mechanics (New York: Wiley), **(2002)**.
[12] J. Stigler, “Analytical velocity profile in tube for laminar and turbulent flow”, Engineering MECHANICS, vol. 21, no. 6, **(2014)**, pp. 371-379.
[13] K. Ramasubramanian and M. Sriram, “A comparative study of computation of lyapunov spectra with different algorithms”, Physica D: Nonlinear Phenomena 139, **(2000)**, pp. 72-86.
[14] M. Romera, G. Pastor, A. Martin, A. B. Orue, F. Montoya and M.-F. Danca, Breaking points in quartic maps, International Journal of Bifurcation and Chaos, vol. 25, **(2015)**, 1550051.
[15] M. J. P. R. H. Landau and C. C. Bordeianu, Computational Physics, “Problem Solving with Computers”, 2nd Revised and Enlarged edn (Wiley-VCH Verlag GmbH & CoKGaA, Weinheim), **(2007)**.
[16] Y. Khmou, “A case study in bifurcation theory”, International Journal of Modern Physics C 28, **(2017)**, 1750104.
[17] S. C. Phatak and S. Suresh Rao, “Logistic map: A possible random-number generator”, Physical Review E, vol. 51, no. 4, **(1995)**.

