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A Comparative Study of Entropy Metrics

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Abstract

We present in this paper, a brief review of the entropy metrics applied on lattice with binary values, the expressions of the entropy functions are described based on the density which is the average value of the lattice where each site can either be occupied by one particle or empty, this methodology is a fundamental concept for modelling the traffic flow of particles based on one dimensional geometry. We present several functions including Shannon entropy, Boltzmann, Rényi and Tsallis entropies, this comparative study is illustrated using computer simulations where we show that the variations of these metrics are described by two phases with critical density of 0.5. In the second part, we explain the link between the density and a particular case of normalized traffic model. Next we present an application of the information entropy in generating random binary sequence.

Keywords: *entropy, lattice, one dimensional, traffic flow, Greenshield's model*

1. Introduction

In the theory of dynamical systems [1, 2], the variations of the states of an ensemble of particles are described by differential equations that describe the dynamics of the system of particles such as non linear phenomena. In the case of discrete dynamical systems, precisely for one dimensional and first order types, the modeling is based on map function of the form $x_{n+1} = f(x_n, \mu)$ which is first order iterative equation relating the state of the system to the previous one using a control parameter μ . The state of system is represented using a convenient phase space such as the position-impulsion diagrams. The verification of the dynamics of the system that consists of an ensemble of particles is performed using computer simulations to study the different properties and phases. The theory of dynamical systems has several applications in different fields including for example biology [3], engineering [4] and economics [5].

The functions describing a given dynamical system are in general continuous, and for the purpose of computer modelling, the discrete values are used where the variables of the system are integers which is the case for example in traffic flow modeling using cellular automaton models [6-8]. Indeed, the traffic flow engineering [9, 10] is a dynamical system that relies on the properties, characteristic and dynamics of the vehicles. The mechanical, kinematic and geometrical configurations of traffic system have an impact on the global behavior of traffic. Indeed, it easy to observe that the average speeds of cars have an impact on the properties of traffic including the flow which is the number of passing vehicles per period of time and the density which is the number of cars per length. From this description, we present in this paper a basic review

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of entropy functions and their applications in some cases of traffic flow which is based on a lattice with periodic boundary conditions and binary values (1,0) where each site can either be occupied by one vehicle or empty. We compare the variations of several metrics including Shannon [11], Boltzmann [12] and collision [13] entropies with respect to the density which is the average value of the lattice. We present the necessity to consider indiscernible vehicles to obtain a reasonable result for the Boltzmann entropy comparatively to other metrics. In the last part, we exlpain the link between the density and the normalized Greenshield's traffic model and we discuss an application of the entropy function for generating binary sequences.

2. Maximum Entropy Principle

We consider a dynamical system of traffic flow with periodic boundary conditions, the properties of this system can be derived using microscopic or macroscopic concepts, the first approach is based on studying the dynamics of individual cars and deriving the global parameters of the system including for example the average velocity and the flow. In this part, we focus on the description of the positions of the vehicles using entropy function which is of great interested in information theory [11] and statistical mechanics [14]. Before beginning the description of the different configurations of the positions of vehicles, we present a fast review of the entropy function. We consider a closed system with total number of microstates Ω, the occurrence of microstate $i \in \{1,...,\Omega\}$ is given by probability p_i with normalization condition:

$$\sum_{i=1}^{\Omega} p_i = 1 \tag{1}$$

By definition, the entropy of the system is positive and logarithmic function h of the probabilities p_i, it is given by the expression:

$$h = -\sum_{i=1}^{\Omega} p_i \log(p_i) \tag{2}$$

Given a microstate i with probability $p_i = 1$ and consequently $p_{j \neq i} = 0$, the entropy is zero which constitute its lower bound, the upper bound is obtained by searching for the maximum value using the normalization condition, the problem can be solved using Lagrange multipliers, let us define the function $L(h,\lambda)$ as follows:

$$L(h,\lambda) = -\sum_{i=1}^{\Omega} p_i \log(p_i) - \lambda\left(1 - \sum_{i=1}^{\Omega} p_i\right) \tag{3}$$

Where λ is Lagrange multiplier, the maximum of L corresponds to the following condition of the first order derivation with respect to p_i:

$$\frac{\partial L(h,\lambda)}{\partial p_i} = \log(p_i) + 1 + \lambda = 0 \tag{4}$$

The expression of the probability is therefore $p_i = e^{-1-\lambda}$, using the normalization condition, the values of the probabilities that maximize the entropy h are given by:

$$p_i = \frac{1}{\Omega} \tag{5}$$

h is maximal when all the microstates have the same probability, where the value of the upper bound of the entropy is obtained by substituting the expressions of p_i as follows:

$$Max\{h\} = -\sum_{i=1}^{\Omega} \frac{1}{\Omega} \log(\frac{1}{\Omega}) = \log(\Omega) \tag{6}$$

Based on this result, we present in the following part, the application of the information entropy function on a system of uniform and periodic traffic flow.

3. Information Entropy

In statistical mechanics [14], the microcanonical ensemble is applied to a system where the number of particles N is constant, similarly we consider, in traffic flow problem, a system with periodic boundary conditions and fixed number cars N , the circuit is discretized into L sites where each site can be empty occupied by one vehicle. Given the condition $0 \le N \le L$, the density of the road is $\rho = N / L$, another interpretation of the density is the probability of presence of cars or the probability of occupation, consequently the probability of absence of cars is $1-\rho$, the presence and absence of cars are denoted by $(1,0)$ respectively, the probabilities are given by:

$$\begin{cases} p(1) = \rho = \dfrac{N}{L} \\ \\ p(0) = 1-\rho = \dfrac{L-N}{L} \end{cases} \tag{7}$$

The two probabilities are linear functions of the variable N such that there exists a value of N where they are equal. This value is $L/2$ where the number of occupied sites equal the number of empty sites, this value corresponds to the average value of the density which can be computed in continuous form as:

$$< \rho >= \frac{1}{\rho_{\max} - \rho_{\min}} \int_{\rho_{\min}}^{\rho_{\max}} \rho d\rho = \int_0^1 \rho d\rho = \frac{1}{2} \tag{8}$$

In discrete form, the average value of the density can also be computed using the variable N that takes $L+1$ different variables as follows:

$$< \rho >= \frac{1}{L(L+1)} \sum_{N=0}^{L} N = \frac{1}{2} \tag{9}$$

By studying the problem using information entropy, we can map the different values of number of cars into single function h where its maximum corresponds to the average value $< \rho >$, we can define h as a function of variable N :

$$h = -\frac{N}{L} \log\left(\frac{N}{L}\right) - \left(\frac{L-N}{L}\right) \log\left(\frac{L-N}{L}\right) \tag{10}$$

When $N = L/2$, the maximum value is $Max\{h\} = \log(2)$. If we consider h to be a function of density ρ, further simplification of the above expression yields the following result:

$$h(\rho) = \rho \log\left(\frac{1-\rho}{\rho}\right) - \log(1-\rho) \qquad (11)$$

Given the probability functions $p(1)$ and $p(0)$ with respect to N, the abscissa of the intersection point corresponds to the maximum of the entropy h as presented in Figure 1 where the computation is carried out using the convention $0\log(0) = 0$.

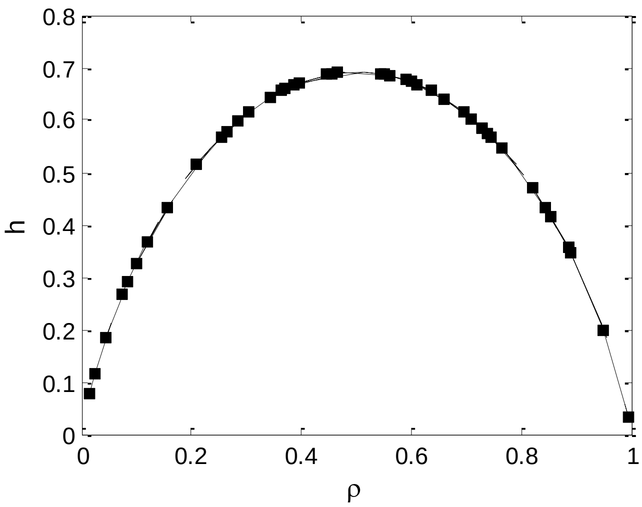

Figure 1. Variation of the Information Entropy h w.r.t Density ρ for Lattice, $L = 200$

For two particular cases of $N = 0$ and $N = L$, the information of the system is maximal such that $h = 0$ which represents the cases of empty and full road where the latter case corresponds to the average velocity of the cars $<v>= 0$. If $N = L/2$, the information about the system is minimal where a site can either be empty or occupied with probability of $p = 0.5$. By introducing the Boltzmann entropy [12], we demonstrate mathematically in the next part, that $N = L/2$ corresponds to the maximum of entropy.

4. Boltzmann Entropy

In micro canonical ensemble, an isolated system consisting of N particles with bounded energy is characterized by Boltzmann entropy [12], which is a function of the number of accessible microstates Ω, it is given by:

$$s = k\log(\Omega) \qquad (12)$$

Where k is the Boltzmann constant, the average energy of the system is given as:

$$<E>= \frac{1}{N}\sum_{i=1}^{N} p_i E_i = \sum_j p_j E_j \qquad (13)$$

Where E_i is the energy of the i^{th} particle and E_j is the j^{th} possible value of the energy with probability p_j. It is clear that if $p_j = \Omega^{-1}$ the entropy s is maximal. Different properties of the system are derived from the function s such as internal energy, pressure, volume and so on. In our problem, we consider that $k = 1$, we describe the system with L sites as the energy levels where the ground state is represented by the first site and maximum allowed energy corresponds to the last site L. Similarly, the particles are equivalent to individual vehicles, so the problem of associating the energy levels to the particles is equivalent to the problem of placing the cars on discrete lattice with periodic boundary conditions. From dynamical viewpoint, given N cars, if we observe the system, we can identify different possibilities of the positions of the cars on the L sites. In the first case we consider that all the vehicles are identical with the same mass, geometry and dynamics, in other words they are indiscernible, the number of microstates is given by:

$$\Omega_i = C_L^N = \frac{L!}{(L-N)!N!} \tag{14}$$

For particular case of empty lattice $N=0$ we have $\Omega_i = 1$ which is also the same in the case of full road $N=L$ where the computation is based on the convention $0!=1$. To analyze the variation of Ω_i as a function of $N=0,1,...,L$, we consider an example of $L=100$. The maximum value corresponds to $N=L/2$ which is equivalent to $\rho=0.5$, the variation of the function C_L^N is characterized by narrow peak located in interval of approximately $\rho=[0.4,0.5]$. For other values of the density, the function appears as constant and equals zero due to the large value of the peak that reaches 10^{28} as illustrated in Figure 2.

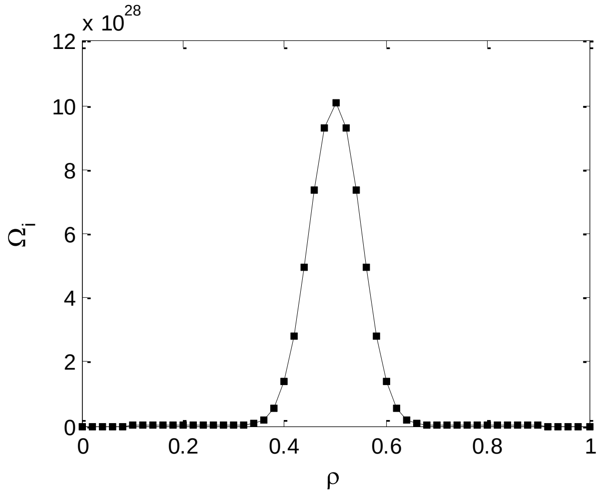

Figure 2. Variation of the Number of Microstates Ω_i for Lattice of $L=100$ Sites with Indiscernible Vehicles

To qualitatively evaluate the variation of Ω_i, we use the logarithm function $s = k\log(\Omega_i)$, thus we obtain a similar expression to the Boltzmann entropy with $k=1$. For this new expression, the equilibrium corresponds to the average value of the density which is in harmony with the obtained result using information entropy as illustrated Figure 3.

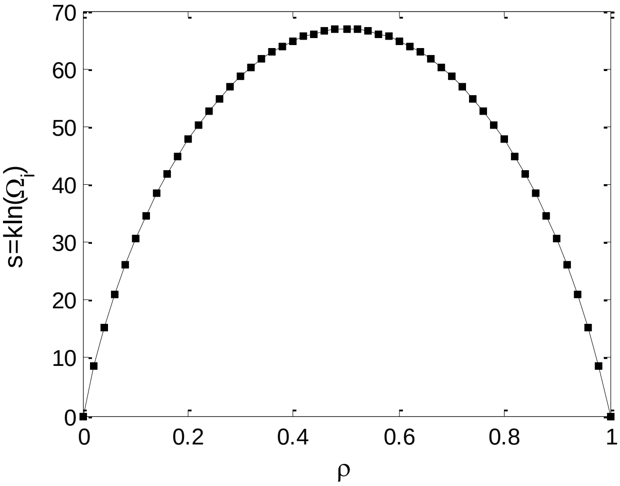

Figure 3. Variation of the Boltzmann Entropy $s = k\log(\Omega_i)$ for Lattice of $L = 100$ Sites with Indiscernible Vehicles

Besides the case of indiscernible vehicles, we consider a hypothesis of non identical vehicles, the most common example is a system consisting of mixture of vehicles with different lengths, a basic configuration is a set of short vehicles which can occupy one site while long vehicles occupy two sites, in this case the number of possibilities changes accordingly. We illustrate the difference between indiscernible and discernible vehicles in terms of the number of microstates by considering an example of $N = 2$ vehicles and lattice of $L = 4$ sites as given in Figure 4.

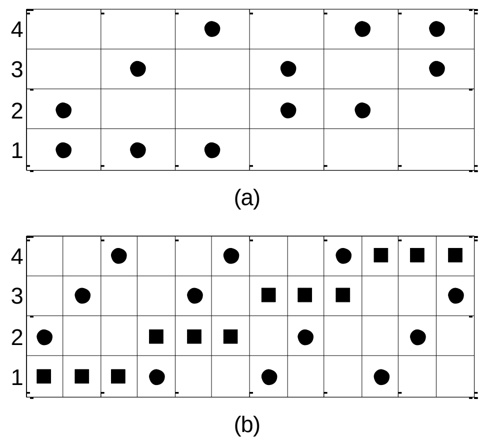

Figure 4. Representation of the Possible Configurations of Placing $N = 2$ Vehicles on Lattice of $L = 4$ Sites, (a) Indiscernible Vehicles, (b) Discernible Vehicles

It is clear that in this example, the case of discernible vehicles doubles the number of the microstates comparatively to the case of indiscernible vehicles where for the first case we have 12 microstates. Generally the numbers of microstates of both cases are related by the factor $N!$, in the case of discernible vehicles, we have:

$$\Omega_d = A_L^N = \frac{L!}{(L-N)!} \tag{15}$$

Where we have the fundamental property between the two ensembles:

$$\Omega_d = \frac{\Omega_i}{N!} \tag{16}$$

Contrarily to the case of Ω_i, the variation of the ensemble Ω_d as function of N yields different results where the maximum number of microstates corresponds to $N = L$. For densities less than $0.5, \Omega_d$ increases similarly to Ω_i, but for densities larger than 0.5, the ensemble continues to increase and reaches its maximal value $L!$which corresponds to $\rho = 1$. The variation of the microcanonical ensemble is presented in Figure 5 using the value $L = 100$ where the logarithmic scale of base 10 is used for the y axis.

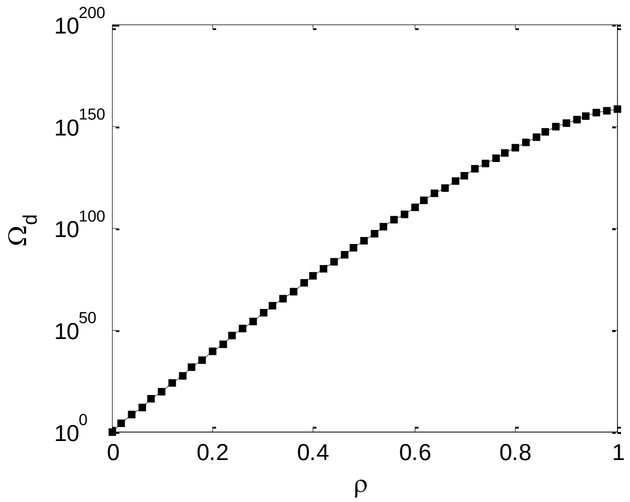

Figure 5. Variation of the Number of Microstates Ω_d for Lattice of $L = 100$ Sites with Discernible Vehicles

From computational viewpoint, among the well known methods for simulating the dynamics of traffic flow is Nagel Schrekenberg model [15] based on parallel update of positions and velocities of individual vehicles, it is cellular automaton model where global parameters are extracted after sufficient number of time updates which can be approximated by $T = 10L$ for lattice with length L as described in [15]. In the simulation steps, the velocities of individual vehicles are decreased randomly by unity with probability p, in the case $p = 0$ the system is deterministic where the fluctuations of the velocities are not included in the system. The Characteristics and the phases of the system are derived from the fundamental diagram of flow and density (ρ, j), for non-deterministic case $p = 0.25$, the fundamental diagram using mean field theory and simulation was reported in [16], where the curve of the flow j is symmetrical with respect to the line $\rho = 0.5$ in the case of maximum velocity $v_{\max} = 1$, this symmetry breaks down for other cases $v_{\max} > 1$. Next, we consider Rényi entropy [17] which is a generalization of Shannon entropy, it is defined by set of probability coefficients p_i for $i = 1, ..., N$ and real number $0 < q < +\infty$ and $q \neq 1$,as the following:

$$h(p,q) = \frac{1}{1-q} \log \sum_{i=1}^{N} p_i^q \tag{17}$$

A special case of this entropy is defined by a parameter $q = 2$ and is called collision entropy [13] which is used in quantum information theory. Given our lattice of L sites with probability $p(1) = \rho$, we can define a simplified expression of the collision entropy as:

$$h(\rho, q = 2) = -\log(2\rho^2 - 2\rho + 1) \tag{18}$$

In Figure 6, we compare the shapes of the information and collision entropy functions where we remark a minimal difference of their variations for the two phases separated by critical value of $\rho = 0.5$.

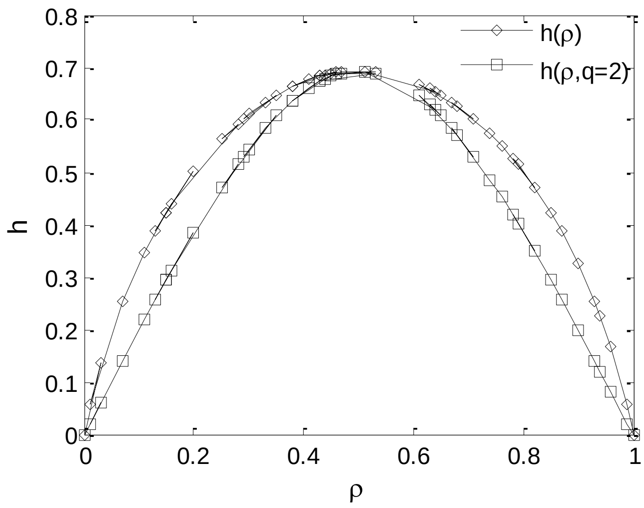

Figure 6. Comparison between Shannon Entropy h and Collision Entropy $h(\rho, q = 2)$ for Lattice of $L = 100$ Sites

The fourth model that we include in this study is Tsallis entropy [18] used in statistical mechanics, which is used to describe a classical system with states described by the probabilities p_i for $i = 1, \ldots, N$, the Tsallis entropy is defined with parameter $\alpha > 0$ and $\alpha \neq 1$ as the following:

$$h(\alpha) = \frac{k}{\alpha - 1}\left(1 - \sum_{i=1}^{N} p_i^{\alpha}\right) \tag{19}$$

Where the constant k=1. Based on the configuration of lattice with binary values, we can rewrite the function as:

$$h(\alpha) = \frac{1}{\alpha - 1}\left(1 - \rho^{\alpha} - (1 - \rho)^{\alpha}\right) \tag{20}$$

The index of the maximum is obtained by solving the equation:

$$\frac{\partial h}{\partial \rho} = \frac{1}{\alpha - 1}\left(\alpha(1 - \rho)^{\alpha - 1} - \alpha\rho^{\alpha - 1}\right) = 0 \tag{21}$$

Which gives similar result $\rho = 0.5$ comparatively to the other entropy functions. Using different values of the parameter α, we present in Figure 7, the variation of the Tsallis entropy using the same parameters of the simulation as the previous cases.

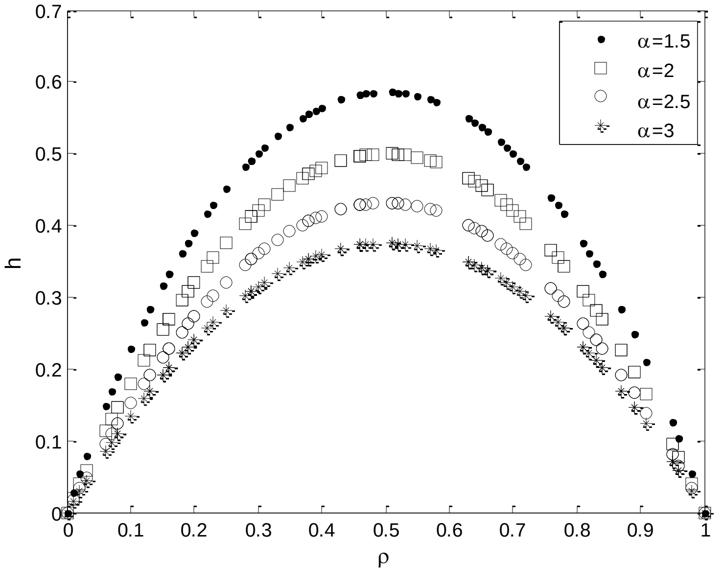

Figure 7. Variation of Tsallis Entropy for Lattice of $L = 100$ sites, with Different Values of the Parameter α

We remark that the different functions have the same critical density, the magnitude of the function decreases as α increases, given $h(\alpha, \rho = 0.5)$, we can deduce the maximum value of the entropy as a function of the parameter α, the maximum value is equal to:

$$Max\{h\} = \frac{1}{\alpha - 1}\left(1 - 2^{1-\alpha}\right) \quad (22)$$

Further analysis can be carried out to illustrate the variation of the entropy metrics with respect to the velocity instead of the positions of the particles. In this case, the dynamics of the system are included where it is necessary to consider a model of how the average velocity of the particles changes with respect to the density $< v(\rho) >$. For simple explanation, we consider a deterministic model. At low densities, all the particles are moving with the allowed maximum velocity. In the congested phase, the flow decreases and deceleration is mandatory where the different values of the velocity $\{0,1,...,v_{\max}\}$ appear in the system. In this phase, the entropy of the velocity of the N particles increases where it reaches its maximal value such that all the values have the same probability. One of the basic model of the variation of the velocity as a function of density is the Greenshield's model [19, 20] where the variation of the velocity is linear, at low density the average velocity is maximal and at $\rho \to 1$ we have $< v >= 0$, if we reconsider again the probability of absence of the particles as explained in equation (7) which is given by $p(0) = 1 - \rho$, we already remark that it corresponds to the normalized Greenshield's model and the corresponding flow can be deduced using the relation:

$$j = p(0)\,p(1) \quad (23)$$

Where by solving the equation $\partial j / \partial \rho = 0$, we obtain the critical density that is also $\rho = 0.5$ which separates free and congested phases.

The applications of the entropy functions span several fields including communication theory [11] and chaos characterization of discrete dynamical systems [21]. For example, the logistic map [22] is non linear discrete dynamical system which is given by first order iterative equation $x_{n+1} = \mu x_n(1-x_n)$ with growth parameter $\mu \in [0,4]$ where the range of the sequence is $x_n \in [0,1]$. For a value $\mu = 4$, the logistic map can be used to generate random numbers [21]. For binary sequence, the generated sequence from the logistic map can be transformed into binary values which can be used for encryption procedures [23] using a threshold such that the randomness must be optimized. The two values of the sequence must have the same probability. To obtain the optimal threshold, the entropy function can be used by searching for a threshold that maximizes the entropy function [24].

5. Conclusion

In this paper, we have studied the application of several entropy metrics on a lattice with binary values, where each site can either be occupied by one particle or empty, this condition is suitable for modelling the traffic flow of particles using one dimensional geometry. Based on the density variable which is the average value of the lattice, we have presented the expressions of the entropy functions that are used in information theory and statistical mechanics. The study was based on computer simulations where we have shown that the variations of the entropy functions are defined by two phases with critical density of 0.5, such that the number of occupied sites equals the number of empty sites regardless of the geometric repartition of the particles on a lattice. In the second part, we have presented a particular case of Greenshield's model of traffic flow using the density variable. Next, we have discussed the usefulness of the entropy function for generating random binary sequences.

Appendix

1. In this part, we present the program to compute the number of microstates for the case of indiscernible particles, the outputs of the program are the ensemble Ω_i and the density ρ.

```
L=100;
c=1;
Omega=zeros(1,L+1);
rho=zeros(1,L+1);
for N=0:L
    rho(c)=N/L;
    Omega(c)=(factorial(L)/(factorial(N)*factorial(L-N)));
    c=c+1;
end
```

2. The presented formalism of the entropy metrics for a lattice with binary values, is related to binomial random variable. Given a lattice with length L, and number of particles N with corresponding density ρ, the binomial probbaility is given by the relation $P(N) = C_L^N \rho^N (1-\rho)^{L-N}$, with normalisation condition:

$$\sum_{N=0}^{L} P(N) = \sum_{N=0}^{L} C_L^N \rho^N (1-\rho)^{L-N} = 1 \tag{24}$$

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